PROPER FORCING AXIOM AND SELECTIVE SEPARABILITY

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Abstract. We continue the study of Selectively Separable (SS) and, a game-theoretic strengthening, strategically selectively separable spaces (SS^+) (see [1]). The motivation for studying SS^+ is that it is a property possessed by all separable subsets of $C_p(X)$ for each σ-compact space X. We prove that the winning strategy for countable SS^+ spaces can be chosen to be Markov. We introduce the notion of being compact-like for a collection of open sets in a topological space and with the help of this notion we prove that there are two countable SS^+ spaces such that the union fails to be SS^+, which contrasts the known result about SS spaces. We also prove that the product of two countable SS^+ spaces is again countable SS^+. One of the main results in this paper is that the proper forcing axiom, PFA, implies that the product of two countable Fréchet spaces is SS, a statement that was shown in [1] to consistently fail. An auxiliary result is that it is consistent with the negation of CH that all separable Fréchet spaces have π-weight at most ω_1.

1. Introduction

The notion of selective separability (or SS) was introduced by Marion Scheepers [14] and is defined as follows.

Definition 1.1. A space X is called selectively separable (or SS) if for each sequence $\{D_n\}_n$ of dense sets, there is a finite selection $E_n \subset D_n$ such that $\cup_n E_n$ is dense.

Many authors now prefer the terminology M-separable for this notion (e.g. see [4, 12]). While studying selective separability, we were interested to explore the topological game related to selective separability.

Definition 1.2. A space has the property SS^+, if player II has a winning strategy for the natural game: player I picks a dense set D_n; player II picks a finite set $E_n \subset D_n$. Player II wins if $\bigcup_n E_n$ is dense.

Gruenhage has asked if player II would always have a Markov strategy in each SS^+ space. A strategy is Markov if it only depends on which move it is and the other player's previous move.

Since SS seems to have arisen in the study of fan tightness in the spaces of the form $C_p(X)$, it is natural to introduce the idea of strategic fan tightness. We observe that if a space X is σ-compact then $C_p(X)$ has strategic fan tightness, so all the separable subsets will be SS^+. Pursuing the duality between the properties of a space X and the base properties of $C_p(X)$, we introduce the idea of a collection of open subsets from a space being compact-like and notice that the property of

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being SS^+ is not finitely additive while it is productive in case of countable spaces. In pursuit of an answer to Gruenhage's question, we are able to show that if an SS^+-space is countable then it has a Markov strategy for being SS^+.

In [1], it is shown that separable Fréchet spaces are SS, and it was established from CH that there can be two countable Fréchet spaces whose product is not SS. In this paper we prove that, with the assumption of PFA the product of two countable Fréchet spaces is SS. Also we find that a still stronger statement is true in the Cohen model, namely that a countable Fréchet space has a π-base of cardinality at most ω_1. It is shown in [14] that each countable space with π-weight less than $\mathfrak{d}$ is selectively separable, which immediately shows that in the Cohen model the product is SS.

2. On Strategic Fan Tightness and SS^+

Let us start this section by recalling the definition of countable fan tightness of a topological space X.

Definition 2.1. [14] A space S has countable fan tightness at x if for each sequence $\langle A_n : n \in \omega\rangle$ of subsets of S each with x in the closure, there is a sequence of finite sets $\langle a_n : n \in \omega\rangle \in \Pi_n[A_n]^{<\omega}$ such that x is in the closure of $\bigcup_n a_n$.

We let *countable dense fan tightness* refer to the property we get by restricting each A_n to be dense. Using this definition for countable fan tightness we introduce the natural game, namely, strategic fan tightness at a point defined as:

Definition 2.2. A space S has strategic fan tightness at a point $x \in S$ if Player II has winning strategy for the following game:

- Player I plays A_n with $x \in \overline{A}_n$.
- Player II selects $a_n \in [A_n]^{<\omega}$.
- Player II wins if $x \in \overline{\cup_n a_n}$

This definition leads to the following immediate Lemma.

Lemma 2.3. *A space X is SS (SS^+) if it is separable and has (strategic) countable dense fan tightness at each point.*

We recall the following definition,

Definition 2.4. A space X is Menger if given a sequence $\langle U_n : n \in \omega\rangle$ of open covers of X, there is a sequence $\langle W_n\rangle_n \in \Pi_n[U_n]^{<\omega}$ such that $\bigcup_n W_n$ is again a cover.

The next result is due to Arhangelski [2],

Theorem 2.5. *$C_p(X)$ has countable fan tightness if and only if X^k is Menger for all $k \in \omega$.*

Our investigation is inspired by the connection between strategic fan tightness in $C_p(X)$ and the σ-compactness of X. We include this next result from [1, 3.6], for motivation and the reader's convenience.

Proposition 2.6. *If X is σ-compact then $C_p(X)$ has strategic fan tightness at each point; and so separable subsets of $C_p(X)$ are SS^+.*

Proof. Since $C_p(X)$ is homogeneous, it suffices to show that $C_p(X)$ has strategic fan tightness at the constant zero function $\underline{0}$. Let $\{X_k : k \in \omega\}$ be an increasing chain of compact sets which cover X. We recall that, $C_p(X)$ is simply a subspace of $\mathbb{R}^X$; where a basic open subset (neighborhood of $f \in C_p(X)$) is, $[f \restriction \{x_i : i < n\}; \epsilon] = \{g \in C_p(X) :\mid g(x_i) - f(x_i) \mid < \epsilon \text{ for } i < n\}$. Now player I chooses $A_n \subset C_p(X)$ with $\underline{0} \in \overline{A_n}$. Let $U_n = \{(a^{-1}(-\frac{1}{n}, \frac{1}{n}))^k : k \leq n \text{ and } a \in A_n\}$. We claim that U_n contains an open cover of $(X_k)^k$ for each $k \leq n$. Indeed, for any $k \leq n$ and $H \in (X_k)^k$, $[\underline{0} \restriction H; \frac{1}{n}]$ is a neighborhood of $\underline{0}$ and so must intersect A_n. Thus, as required, there is some $a \in A_n$ such that $H \in (a^{-1}(-\frac{1}{n}, \frac{1}{n}))^k$. Now, since each $(X_k)^k$ is compact, player II may select a finite $e_n \subset A_n$ so that the finite subcollection $W_n = \{(a^{-1}(-\frac{1}{n}, \frac{1}{n}))^k : k \leq n \text{ and } a \in e_n\}$ is a cover of $(X_k)^k$ for each $k \leq n$. Now we are left to show that $\underline{0} \in \overline{\bigcup_n e_n}$. To show that, let us fix any k, $\{x_i : i < k\} \subset X$ and $\epsilon > 0$. We need to show there is an $a \in \bigcup_n e_n$ such that $a \in [\underline{0} \restriction \{x_i : i < k\}; \epsilon]$. Choose $n \geq k$ so large that $\{x_i : i < k\} \subset X_n$ and $\frac{1}{n} < \epsilon$. It follows then that there is an $a \in e_n$ such that $\langle x_i : i < k\rangle \in \left(a^{-1}(-\frac{1}{n}, \frac{1}{n})\right)^k$; and therefore, $a \in [\underline{0} \restriction \{x_i : i < k\}; \epsilon]$ as required. □

As mentioned previously, Gruenhage asked whether there is always a Markov strategy in SS$^+$ spaces. In such a case let us say that the space is Markov SS. We show that there is always a connection if the space is countable.

When studying SS or SS$^+$ for the spaces like $S = C_p(X, 2) \subset 2^X$, the role of X can be thought of as enumerating the base for S, and a compact subset of X plays a crucial role in SS$^+$. Keeping that in mind we define the notion of a subcollection of open sets being compact-like in a space, which we define as follows:

Definition 2.7. Suppose S is a space and $\mathcal{C}$ is a collection of (open) subsets of S. We say that $\mathcal{C}$ is compact-like, if for all dense $D \subset S$, there is a finite $e \subset D$ such that $e \cap C \neq \emptyset$ for all $C \in \mathcal{C}$.

It is immediate from the definition, that if $\mathcal{E}$ is a family of finite subsets of a space S such that each dense set contains a member of $\mathcal{E}$, then any family $\mathcal{C}$ of open sets which meets every member of $\mathcal{E}$ will be a compact-like family.

The notion of σ-compact-like is defined as follows:

Definition 2.8. A space S is σ-compact-like, if the topology τ related with S is σ-compact-like, that is, if τ can be written as countable union of compact-like open subcollections of τ.

Lemma 2.9. *If a space S is σ-compact-like, then S has a Markov strategy for being SS$^+$, i.e., S will be Markov SS.*

Proof. Since S is σ-compact-like, it has a σ-compact-like base, say $\mathcal{B}$. Let $\mathcal{B} = \bigcup_n \mathcal{B}_n$, where $\langle \mathcal{B}_n \rangle_n$ is an increasing family and each of them is compact-like. So for each dense $D \subset S$ and each n, there exists a finite $e_n \in D$ such $e_n \cap B \neq \emptyset$ for each $B \in \mathcal{B}_n$. We show that this selection, $e_n \subset D$ at stage n is the desired Markov strategy for Player II. Indeed, let, at stage n, player I plays A_n, where A_n is dense in S. Player II will choose a finite set $e_n \subset A_n$ as above, i.e. so that $e_n \cap B \neq \emptyset$ for all $B \in \mathcal{B}_n$. It is immediate that $\overline{\bigcup_n e_n}$ is dense since it meets every member of the base $\mathcal{B}$. □

Also we have the next result,

Theorem 2.10. *If X is Markov SS, then S is σ-compact-like.*

Proof. Let $\mathcal{D}$ be the collection of all dense subsets of X. Since X is Markov SS there is a winning strategy σ with domain $\mathcal{D}\times\omega$, where, for each $(D,n)\in\mathcal{D}\times\omega$, $\sigma(D,n)$ is a finite subset of D. Now let us consider the collection $\mathcal{C}_n=\{C\in\mathcal{B}:$ for $D\in\mathcal{D}, C\cap\sigma(D,n)\neq\emptyset\}$. From the definition of $\mathcal{C}_n$, it is clear that each of them is compact-like, so $\bigcup_n\mathcal{C}_n$ is σ-compact-like. So we just need to prove that the collection $\bigcup_n\mathcal{C}_n$ is a base. To show this, let $x\in X$ and U be any open set such that $x\in U$. If no member of $\mathcal{C}_n$ is contained in U then for some $D_n\in\mathcal{D}$, $\sigma(D_n,n)$ misses U. If we can find D_n for each n, then the fact that $\bigcup_n\sigma(D_n,n)$ misses an open set, contradicts that it is to be a dense union. Therefore $\bigcup_n\mathcal{C}_n$ is a σ-compact-like base. □

Theorem 2.11. *If a space X is countable and SS$^+$ then it is Markov SS.*

Proof. The space is SS$^+$, so there is a SS$^+$ strategy σ on X. Let $\mathcal{D}$ denote the family of dense subsets of X. Our assumption on σ is that it is a function with domain consisting of finite sequences $\langle D_i : i\leq n\rangle$ from $\mathcal{D}$, satisfying that $\sigma(\langle D_i : i\leq n\rangle)$ is a finite subset of D_n and, for all infinite sequences $\langle D_i : i\in\omega\rangle$ from $\mathcal{D}$, the sequence $\{\sigma(\langle D_i : i\leq n\rangle) : n\in\omega\}$ of finite subsets of X will have dense union.

We now show that X is σ-compact-like. We will recursively define a tree T consisting of finite sequences of finite subsets of X which result from partial plays of the game following the strategy σ. Thus, if $t\in T$ there is an integer $\ell=dom(t)$, and for each $i<\ell$, $t(i)$ is a finite subset of X. Furthermore, $t\in T$ if and only if there is a fixed sequence $\langle D^t_i : i<\ell=dom(t)\rangle\in\mathcal{D}^\ell\rangle$ such that for each $i<\ell$, $t(i)=\sigma(\langle D^t_j : j\leq i\rangle)$. An important additional assumption is that if $t\subset s$ are both in T, then for $i\in dom(t)$, $D^t_i=D^s_i$.

We begin with the empty sequence as an element of T. It follows easily that for each $t\in T$, $\mathcal{E}_t=\{\sigma(\langle D^t_0,\ldots,D^t_{dom(t)-1},D\rangle) : D\in\mathcal{D}\}$ is a family of finite subsets of X satisfying that every dense set includes one. Let $dom(t)=\ell$, and for each $e\in\mathcal{E}_t$, we have that $s_e\in T$ where $dom(s_e)=\ell+1$, $s_e\supset t$ and $s_e(\ell)=e$. In addition, for $s=s_e$, $D^s_i=D^t_i$ for $i\in dom(t)$, and D^s_ℓ is chosen to be any $D\in\mathcal{D}$ such that $\sigma(\langle D^s_i : i\leq\ell\rangle)=e$. Therefore, the collection $\mathcal{C}_t$ (or $\mathcal{D}_{\mathcal{E}_t}$) is compact-like, where an open subset U of X is in $\mathcal{C}_t$ if and only if it meets every member of $\mathcal{E}_t$.

We show that every non-empty open set is in $\bigcup_{t\in T}\mathcal{C}_t$; thus showing that the topology on X is σ-compact-like. Assume otherwise, and assume that $U\notin\mathcal{C}_t$ for all $t\in T$. By a simple recursion, choose an increasing chain $\{t_n : n\in\omega\}$ in T so that $U\cap t_{n+1}(n)$ is empty for each n. It follows easily that $\langle D^{t_{n+1}}_n : n\in\omega\rangle$ is a play of the game that the strategy σ fails to defeat by virtue of the fact that the union of Player II's play will miss U. □

The above connections between countable SS$^+$-spaces and the property of being σ-compact-like is instrumental in our approach to discovering that the union of two SS$^+$ spaces need not be SS$^+$. This is quite surprising since it was shown in [11, 12] that the property SS is finitely additive. In [1], we produced an example of a space being SS but not SS$^+$. By the next result we now have another example of an SS$^+$ space which is not SS, namely the union of the two SS$^+$ spaces.

Theorem 2.12. *There are countable SS$^+$ spaces A, B such that $A\cup B$ is not SS$^+$.*

Proof. For $x \in 2^\omega$, let us define $x^\dagger$ by flipping the first value, i.e., $x^\dagger = \langle 1 - x(0), x(1), x(2), ...\rangle$. Let $Z \subset 2^{2^\omega}$ be defined by,

$$Z = \{z \in 2^{2^\omega} : z(x) \cdot z(x^\dagger) = 0 \ \forall x \in 2^\omega\} = 2^{2^\omega} \setminus \bigcup_{x \in 2^\omega} ([x;1] \cap [x^\dagger;1]),$$

where $[x;i]$ is the basic open neighborhood of a function which takes x to i for $i \in \{0,1\}$. Let $A = C_p(2^\omega, 2) \cap Z$. Since $C_p(2^\omega, 2)$ is SS$^+$, A is SS$^+$. To identify the set B, we first define a new topology $\tau^\dagger$ on 2^ω. Let Q denote the countable dense set of rationals in 2^ω (the sequences that are eventually 0). The basic open sets in $\tau^\dagger$ are of the form $[s]_{\tau^\dagger} = [s] \setminus Q \cup ([s^\dagger] \cap Q)$ for any $s \subset 2^{<\omega} = \bigcup_n 2^n$. It is immediate that this space is just another copy of the Cantor set obtained by a simple permutation on the elements of Q. Now we define $B = C_p((2^\omega, \tau^\dagger), 2) \cap Z$. Again it follows immediately that B is SS$^+$.

We claim that A and B are mutually dense in Z. We show that A is dense in Z and omit the simple modification necessary to show that B is also dense in Z. Let us consider any $l \in \omega$ and let $\cap_{i<l}([x_i;0] \cap [y_i;1])$ be a basic open set meeting Z. Note that since this basic open set does meet Z, we have that $\forall i \neq j$, $y_i^\dagger$ is not equal y_j. To show that this basic open set hits A, we pick m so large that, first, if any of the members of $\{x_i, y_i : i < l\}$ are rationals, they are constant above m, and secondly, any two distinct elements of $\{x_i, y_i, x_i^\dagger, y_i^\dagger : i < l\}$ will differ somewhere below m. Let $a \in C_p(2^\omega, 2)$ be defined so that whenever $t \in 2^m$, $a[t] = 1$ if and only if there is an $i < l$ such that $t = y_i \restriction m$. It is clear that $a \in \cap_{i<l}([x_i;0] \cap [y_i;1])$. To show $a \in Z$, let $x \in 2^{<\omega}$ and $a(x) = 1$. We need to prove that $a(x^\dagger) = 0$. Let $t = x \restriction m$, therefore $a[t] = 1$ and $t \subset y_i$ for some $i < l$. Now if $a(x^\dagger) = 1$, then there must be some $i \neq j < l$ such that $x^\dagger \restriction m \subset y_j$. Of course it now follows that $y_i^\dagger \restriction m = y_j \restriction m$ which contradicts the assumptions that $y_i^\dagger \neq y_j$ for all $i \neq j$, and that m is large enough to distinguish these elements. Therefore A is dense in Z.

As mentioned above, each of A and B are SS^+. We claim that $A \cup B$ does not have σ-compact-like topology. Assume that $\mathcal{B} = \{[x;1] \cap Z : x \in 2^\omega\}$ can be written as countable union of compact-like sets. By the Baire category theorem then, there is an $I \subset 2^\omega \setminus Q$ which is dense in some Cantor basic clopen set $[s]$ such that $I' = \{[x;1] : x \in I\}$ is compact-like. Let us pick any rational $q \in Q \cap [s]$ and let us define the set $D = (A \cap [q;0]) \cup (B \cap [q^\dagger;0])$. Fix any m_q so that q is constantly 0 above m_q. Since the union of the two open sets $Z \cap [q;0]$ and $Z \cap [q^\dagger;0]$ is dense in Z, it follows immediately that D is dense in Z. Now if $d \in D$ then either $d \in A \cap [q;0]$ or $d \in B \cap [q^\dagger;0]$. Since $q \in [s] = \overline{I}$, there is a sequence $\langle x_n \rangle_n \subset I$ converging to q. We show that d is in only finitely many of the sets from the collection $\{[x_n;1] : n \in \omega\}$ and so no finite subset of D can meet every member of the collection I'. Notice that this is equivalent to proving that $d(x_n) = 0$ for all but finitely many n.

First suppose that $d \in A \cap [q;0]$; hence d is continuous with respect to the usual topology on 2^ω. It follows then that there is an $m > m_q$ such that d sends the entire basic open set $[q \restriction m]$ to 0. Since all but finitely many of the x_n's are in $[q \restriction m]$, this completes the proof of the case $d \in D \cap A$. Now suppose that $d \in B \cap [q^\dagger;0]$. Now d is continuous with respect to $\tau^\dagger$. In this new topology, it is easy to see that the sequence $\{x_n : n \in \omega\}$ converges to the point $q^\dagger$. Thus, since $d(q^\dagger) = 0$, it follows again that $d(x_n) = 0$ for all but finitely many n.

Therefore $A \cup B$ is not σ-compact-like, and so, by Theorems 2.11 and 2.10, this space is not SS^+. $\square$

Now we will prove that Markov SS is finitely productive. For that we need the following lemma,

Lemma 2.13. *Let S be any space and $\mathcal{C}$ be any collection of open sets. Then $\mathcal{C}$ is compact-like if and only if for each ultrafilter $\mathcal{U}$ on $\mathcal{C}$, the collection $S(\mathcal{C},\mathcal{U}) = \{s \in S : \mathcal{C}_s = \{C \in \mathcal{C} : s \in C\} \in \mathcal{U}\}$ has non-empty interior.*

Proof. If $S(\mathcal{C},\mathcal{U})$ does not have non-empty interior, then $D = S \setminus S(\mathcal{C},\mathcal{U})$ is dense and therefore for any finite $F \subset D$, $a \in F$ implies $\mathcal{C}_a = \{C \in \mathcal{C} : a \in C\} \notin \mathcal{U}$, so F does not even meet $\mathcal{U}$-many elements of $\mathcal{C}$. Conversely, assume that for each ultrafilter $\mathcal{U}$ on $\mathcal{C}$, $S(\mathcal{C},\mathcal{U})$ has non-empty interior. Let D be any dense subset of S. Then, for each $d \in D \cap \operatorname{int}(S(\mathcal{C},\mathcal{U}))$, $\mathcal{C}_d = \{C \in \mathcal{C} : d \in C\} \in \mathcal{U}$. Now we can see that the collection $\{\mathcal{C}_d\}_{d \in C}$ covers $\beta\mathcal{C}$. Since $\beta\mathcal{C}$ is compact, there are finitely many members $\{d_1, d_2, ..., d_n\}$ from D, such that $\{\mathcal{C}_{d_i} : i \in \{1, 2, ..., n\}$ is a subcover for $\beta\mathcal{C}$. So we have a finite set $F \subset D$, namely $\{d_1, d_2, ..., d_n\}$, such that $F \cap C \neq \emptyset$ for all $C \in \mathcal{C}$, which shows that $\mathcal{C}$ is compact-like. $\square$

Now we can prove that Markov SS is productive.

Theorem 2.14. *The property of being Markov SS is finitely productive.*

Proof. Let X and Y have σ-compact-like bases $\mathcal{B} = \bigcup_n \mathcal{B}_n$ and $\mathcal{C} = \bigcup_n \mathcal{C}_n$ respectively. We use Lemma 2.13 to show that the collection $\mathcal{A}_n = \{B \times C : B \in \mathcal{B}_n, C \in \mathcal{C}_n\}$ is compact-like. Let $\mathcal{W}$ be any ultrafilter on $\mathcal{A}_n$. We will show that,

$$S(\mathcal{A}_n, \mathcal{W}) = \{(x, y) \in X \times Y : \{(B \times C) \in \mathcal{A}_n : (x, y) \in B \times C\} \in \mathcal{W}\}$$

has non-empty interior. Let us define $\mathcal{W}_0$ and $\mathcal{W}_1$ by $\mathcal{W}_0 = \{W \subset \mathcal{B}_n : {\pi_X}^{-1}(W) = W \times \mathcal{C}_n \in \mathcal{W}\}$ and $\mathcal{W}_1 = \{V \subset \mathcal{C}_n : {\pi_Y}^{-1}(V) = \mathcal{B}_n \times V \in \mathcal{W}\}$. Since $\mathcal{W}$ is an ultrafilter, $\mathcal{W}_0$ and $\mathcal{W}_1$ are both ultrafilters on $\mathcal{B}_n$ and $\mathcal{C}_n$ respectively. We claim that $S(\mathcal{B}_n, \mathcal{W}_0) \times S(\mathcal{C}_n, \mathcal{W}_1) \subset S(\mathcal{A}_n, \mathcal{W})$. Let us choose any $(x, y) \in S(\mathcal{B}_n, \mathcal{W}_0) \times S(\mathcal{C}_n, \mathcal{W}_1)$. Then $x \in S(\mathcal{B}_n, \mathcal{W}_0)$ and $y \in S(\mathcal{C}_n, \mathcal{W}_1)$, hence $(\mathcal{B}_n)_x \in \mathcal{W}_0$ and $(\mathcal{C}_n)_y \in \mathcal{W}_1$. Since $\mathcal{W}$ is an ultrafilter $(\mathcal{B}_n)_x \times (\mathcal{C}_n)_y = {\pi_X}^{-1}((\mathcal{B}_n)_x) \cap {\pi_Y}^{-1}((\mathcal{C}_n)_y) \in \mathcal{W}$. Since $(\mathcal{A}_n)_{(x,y)} \supset (\mathcal{B}_n)_x \times (\mathcal{C}_n)_y$ we have shown that $(x, y) \in S(\mathcal{A}_n, \mathcal{W})$. Therefore $S(\mathcal{A}_n, \mathcal{W})$ contains $S(\mathcal{B}_n, \mathcal{W}_0) \times S(\mathcal{C}_n, \mathcal{W}_1)$. Since $\mathcal{B}_n$ and $\mathcal{C}_n$ are compact-like, both of $S(\mathcal{B}_n, \mathcal{W}_0)$ and $S(\mathcal{C}_n, \mathcal{W}_1)$ have non-empty interior which implies that $S(\mathcal{A}_n, \mathcal{W})$ also has non-empty interior. Therefore $\mathcal{A}_n$ is compact-like. $\square$

Dr. Santi Spadaro has generalized this to prove that Markov SS is even countably productive.

Now we have the following important observation about countable SS^+ spaces.

Proposition 2.15. *The finite product of countable SS^+ spaces is again SS^+.*

The extensive use of ultrafilters does seem somewhat unnatural in dealing with finite products, so we thought it helpful to provide a proof of Theorem 2.14 with more similarity to the standard proof of compactness for the product of two compact spaces. However, we still rely on ultrafilters by using Lemma 2.13. We begin with the following consequence of a collection being compact-like.

Proposition 2.16. *Suppose that $\mathcal{E}$ is a family of finite subsets of a space S with the proper that for all non empty open $U \subset S$, there exists $e \in \mathcal{E}$ such that $e \subset U$. Then for each compact-like collection $\mathcal{C}$ of open subsets of S there exists a finite collection $\mathcal{E}' \subset \mathcal{E}$ such that for all $C \in \mathcal{C}$, there exists $e \in \mathcal{E}'$ with $e \subset C$.*

Proof. We apply Lemma 2.13 as follows. For each ultrafilter $\mathcal{U}$ on $\mathcal{C}$, we have that $S(\mathcal{C},\mathcal{U})$ has non-empty interior. Therefore, there is an $e_{\mathcal{U}} \in \mathcal{E}$ which is contained in $S(\mathcal{C},\mathcal{U})$. Similarly, there is a subcollection $\mathcal{C}_{\mathcal{U}} \in \mathcal{U}$ satisfying that $e_{\mathcal{U}} \subset C$ for all $C \in \mathcal{C}_{\mathcal{U}}$. As in Lemma 2.13, there is a finite set, $\{\mathcal{U}_i : i < n\}$, of ultrafilters on $\mathcal{C}$ such that $\mathcal{C}$ is covered by $\bigcup\{\mathcal{C}_{\mathcal{U}_i} : i < n\}$. It follows immediately, that $\mathcal{E}' = \{e_{\mathcal{U}_i} : i < n\}$ is the desired finite subset of $\mathcal{E}$. □

Proposition 2.17. *If $\mathcal{B}$ and $\mathcal{C}$ are compact-like families of open subsets of X, Y respectively, then $\mathcal{B} \times \mathcal{C}$ is compact-like in $X \times Y$.*

Proof. Let π_X denote the projection map from $X \times Y$ onto X, and fix any dense subset D of $X \times Y$. Let U be any non-empty open set in $X \times Y$. Since $\mathcal{C}$ is compact-like in Y, it is trivial to check that the family $\mathcal{C}_U = \{U \times C : C \in \mathcal{C}\}$ is compact-like in $X \times Y$. Therefore there is a finite $D_U \subset D \cap (U \times Y)$ which meets every member of $\mathcal{C}_U$. Observe that this means that D_U meets $\pi_X(D_U) \times C$ for every $C \in \mathcal{C}$.

Now the family $\mathcal{E} = \{\pi_X(D_U) : \emptyset \neq U \subset X \text{ is open}\}$ (where π_X is the projection onto X) satisfies the hypothesis of Lemma 2.16, and so we may select open sets $\{U_i : i < n\}$ of X so that each $B \in \mathcal{B}$ contains $\pi_X(D_{U_i})$ for some $i < n$. Since D_{U_i} meets $\pi_X(D_{U_i}) \times C$ for all $C \in \mathcal{C}$, this shows that D_{U_i} meets $B \times C$ for all $C \in \mathcal{C}$. Thus $\bigcup_{i<n} D_{U_i}$ is the desired finite set to show that $\mathcal{B} \times \mathcal{C}$ is compact-like. □

3. Products of Fréchet SS spaces

In our previous paper [1], we had shown that, $\mathsf{MA}_{\rm ctble}$ implies that there are countable SS spaces whose product is not SS but we required CH to construct two countable Fréchet spaces whose product was not SS. Of course it is well-known that the Fréchet property itself is not finitely productive. In this section we begin by establishing that $\mathsf{MA}_{\rm ctble}$ is not sufficient by studying Fréchet spaces in the well-known Cohen model. This first result is certainly of independent interest.

Theorem 3.1. *In any model obtained by adding Cohen reals over a model of CH all countable Fréchet spaces have π-weight at most ω_1.*

Proof. We assume our ground model satisfies CH and we consider forcing with $P = Fn(\kappa, 2)$ where κ is some cardinal greater than ω_1. Let $\dot\tau$ be a P-name of a topology on ω so that $X = (\omega, \dot\tau)$ is forced to be a Fréchet space. Let $\dot A_n$ denote the P-name which is forced to be the collection of all sequences converging to n. Let $\theta = 2^{\mathfrak{c}^+}$ and $M \prec H_\theta$ be an elementary submodel such that $M^\omega \subset M$ and $|M| = \omega_1$. Suppose also that X, $\dot\tau$, $\{\dot A_n : n \in \omega\}$ are in M. We will prove that $\dot\tau \cap M$ is forced to be a π-base for $\dot\tau$. This will rely heavily on the fact that the elementary submodel M is closed under ω-sequences. In particular, we have that if G is a P-generic filter, then $V[G \cap M]$ is a submodel of $V[G]$ which will satisfy that the interpretation of $\dot\tau \cap M$ will be a Fréchet topology on ω in which, for each n, the interpretation of $\dot A_n \cap M$ will be the collection of sequences converging to n (see [5, 4.5] for more explanation).

We now proceed by working within the model $V[G \cap M]$ (which we refer to as the ground model) and using that $V[G]$ is obtained by forcing over this model with

$Fn(\kappa \setminus M, 2)$. Through a standard abuse of notation, we may let $\dot{\tau}$ continue to denote the name for the final topology in $V[G]$. Now suppose that $\dot{U}$ is a name of a set forced to be non-empty and a member of $\dot{\tau}$. For each condition p, let $\dot{U}_p^-$ denote the set $\{x \in \omega : p \Vdash x \in \dot{U}\}$. Notice that $\dot{U}_p^-$ is a set in the ground model and is forced by p to be contained in $\dot{U}$. Also, by the elementarity assumptions on M, it also follows that p would force that the ground model closure of $\dot{U}_p^-$ would be contained in the closure of $\dot{U}$.

For a contradiction, let us assume that it is forced that the closure of $\dot{U}$ contains no ground model open set. In particular, by the assumptions on M, we then have that there is a condition p_0 and an integer n such that $p_0 \Vdash n \in \dot{U}$ and for all conditions $p \leq p_0$, $\dot{U}_p^-$ is nowhere dense.

Since $\dot{U}$ is a name of a subset of ω, we may choose a countable set $L \subset \kappa \setminus M$ so that $dom(p_0) \subset L$ and for each $k \in \omega$ and each condition $p \in Fn(\kappa, 2)$, $p \Vdash k \in \dot{U}$ implies $p \restriction L \Vdash k \in \dot{U}$. In effect, $\dot{U}$ is a $Fn(L, 2)$-name, and let $\{p_\ell : \ell \in \omega\}$ enumerate those members of $Fn(L, 2)$ which extend p_0. Since, for each n, $\dot{U}_{p_0}^- \cup \dot{U}_{p_1}^- \cup ..... \cup \dot{U}_{p_n}^-$ is nowhere dense, it follows that, the complement of the closure of this union, D_n, is dense. As mention, [1, 2.9], each countable Fréchet space is SS, so there is a selection $F_n \in [D_n]^{<\omega}$ such that $\bigcup_n F_n$ is dense.

Since the space is Fréchet and $x \in \overline{\bigcup_n F_n}$, there is a sequence $S_x \subset \bigcup_n F_n$ converging to x. By the definition of the D_n's, we have that S_x is almost disjoint from $\dot{U}_p^-$ for each $p \in Fn(L, 2)$ which extends p_0. On the other hand, since S_x converges to x, we have, by elementarity, S_x converges to x in the final model, and so there must be a condition p which forces that S_x is almost contained in $\dot{U}$. This is the desired contradiction. □

Corollary 3.2. In the Cohen model, finite products of countable Fréchet spaces are SS.

Proof. It was shown in [14], that if a space is separable and has π-weight less that $\mathfrak{d}$ then it is SS. Our last theorem shows that in the specified Cohen model, all countable Fréchet spaces have π-weight at most ω_1. So the product will also have π-weight at most ω_1, which is less than $\mathfrak{d}$. Therefore the product is SS. □

The next theorem shows us the same conclusion as before assuming PFA.

Theorem 3.3. *The proper forcing axiom, PFA, implies that products of finitely many countable Fréchet spaces are SS.*

Proof. Let X and Y be countable Fréchet spaces and we assume that their product is not SS. There is no loss of generality to assume that neither X nor Y has isolated points. Let $\{E_n : n \in \omega\}$ be a sequence of dense subsets of $X \times Y$. It is known ([1, 2.7]) and easy to see that it is sufficient to show that each point $(x_0, y_0) \in X \times Y$ is in the closure of the union of some sequence of finite selections. So we fix a point (x_0, y_0). Without loss of generality, we may also arrange that the E_n's are a descending sequence. Let $\mathcal{A}_{x_0} \subset [X]^\omega$ and $\mathcal{B}_{y_0} \subset [Y]^\omega$ be the collection of all sequences converging to x_0 and y_0 respectively. Let $\{x_i : i \in \omega\}$ and $\{y_i : i \in \omega\}$ be enumerations of X and Y respectively. Since there is no harm to shrink the E_n's, we will assume that each E_n is disjoint from the closed nowhere dense sets $\{x_i : i < n\} \times Y$ and $X \times \{y_i : i < n\}$. For each $(A, B) \in \mathcal{A}_{x_0} \times \mathcal{B}_{y_0}$, we may assume there is an m such that $E_m \cap (A \times B)$ is empty, because otherwise there is a suitable selection of finite choices accumulating to (x_0, y_0). To see this, first notice that there must be an n such that (x_0, y_0) is not in the closure of $E_n \cap (A \times B)$. It follows that such an E_n will satisfy that, for some $m > n$, $E_n \cap (A \times B) \subset ((\{x_i\}_{i<m} \times Y) \cup (X \times \{y_i\}_{i<m}))$. Then we choose our $m > n$ by our additional assumption that E_m is disjoint from $\{x_i\}_{i<m} \times Y$ and $X \times \{y_i\}_{i<m}$.

Now we consider the poset P defined by the following: $P = \bigcup_n \Pi_{k<n} E_k$ where P is ordered by set inclusion. Of course the members of P are just finite partial selections from the sequence $\langle E_k : k \in \omega\rangle$ and forcing with P gives rise to a name of a generic selection $\dot{F} = \{p(k) : k \in \omega\}$. Notice also that no x and no y will appear as a coordinate in infinitely many of the pairs $\{p(k) : k \in \omega\}$.

We will prove, using an auxiliary proper poset extending P, that there is a family of ω_1-dense sets which are sufficient to ensure that (x_0, y_0) is forced to be in the closure of $\dot{F}$. Establishing this completes the proof of the theorem since PFA implies there is a filter meeting each of those dense sets. The methodology is borrowed from the theory behind the development of the Open Coloring Axiom.

In the generic extension by P, notice that for any $A \in \mathcal{A}_{x_0}$ and $B \in \mathcal{B}_{y_0}$, we have that $F \cap ((A \times Y) \cap (X \times B)) = F \cap (A \times B)$ is finite (since some E_m misses $A \times B$). For $A \in \mathcal{A}_{x_0}$, let $\tilde{A} = F \cap (A \times Y)$ and, for $B \in \mathcal{B}_{y_0}$, let $\tilde{B} = F \cap (X \times B)$. Let $\mathfrak{X} = \{(\tilde{A}, \tilde{B}) : \tilde{A} \cap \tilde{B} = \emptyset\}$. Now we define $K_0 \subset [\mathfrak{X}]^2$ as follows: $\langle(\tilde{A}_0, \tilde{B}_0), (\tilde{A}_1, \tilde{B}_1)\rangle \in K_0$ if $(\tilde{A}_0 \cap \tilde{B}_1) \cup (\tilde{B}_0 \cap \tilde{A}_1) \neq \emptyset$. The separable metric topology on $\mathfrak{X}$ is defined by the following: for finite subsets u_0, u_1, v_0, v_1 of $X \times Y$, the basic open sets are of the form $[(u_0, u_1), (v_0, v_1)] = \{(\tilde{A}, \tilde{B}) \in \mathfrak{X} : u_0 \subset \tilde{A}, u_1 \cap \tilde{A} = \emptyset, v_0 \subset \tilde{B}, v_1 \cap \tilde{B} = \emptyset\}$. Notice that K_0 is an open set in this topology.

Let $K_1 = [\mathfrak{X}]^2 \setminus K_0$. Since K_0 is open in $[\mathfrak{X}]^2$, then by [6], we can say that either $\mathfrak{X}$ is a countable union of 1-homogeneous sets or there is a proper poset, Q which introduces an uncountable 0-homogeneous set.

First we show that if indeed $\mathfrak{X}$ can not be covered by a countable union of 1-homogeneous sets then we obtain our desired selection F from the E_n's accumulating to (x_0, y_0). In this case then, there exists a P-name $\dot{Q}$ for a proper poset such that $\dot{Q}$ introduces an uncountable 0-homogeneous set. That is, there is a $P * \dot{Q}$-name of a sequence, $\langle(\dot{A}_\alpha, \dot{B}_\alpha) : \alpha \in \omega_1\rangle$ of pairs from $\mathcal{A}_{x_0} \times \mathcal{B}_{y_0}$ so that (it is forced that) $\{(\widetilde{\dot{A}_\alpha}, \widetilde{\dot{B}_\alpha}) : \alpha \in \omega_1\}$ is a K_0-homogeneous subset of $\mathfrak{X}$. It is somewhat routine to verify that there is a family of ω_1-many dense subsets of $P * \dot{Q}$ so that an application of PFA ensures that we obtain an infinite selector F from $\langle E_n\rangle_n$ and a sequence $\{(A_\alpha, B_\alpha) : \alpha \in \omega_1\} \subset \mathcal{A}_{x_0} \times \mathcal{B}_{y_0}$ satisfying that for each $\alpha \neq \beta \in \omega_1$,

$$(F \cap \tilde{A}_\alpha) \cap (F \cap \tilde{B}_\alpha) = \emptyset$$

and

$$F \cap [(\tilde{A}_\alpha \cap \tilde{B}_\beta) \cup (\tilde{A}_\beta \cap \tilde{B}_\alpha)] \neq \emptyset \text{ and is finite.}$$

The above properties are the requirements that the families $\{F \cap (A_\alpha \times Y) : \alpha \in \omega_1\}$ and $\{F \cap (X \times B_\alpha) : \alpha \in \omega_1\}$ form a Luzin gap and so, [13], cannot be mod finite separated in $\mathcal{P}(X \times Y)$. Now we show that if $U \times W$ is a neighborhood of (x_0, y_0), then $U \times W$ meets F – as required. Notice that $U \times Y$ will contain, mod finite, $F \cap (A_\alpha \times Y)$ for all $\alpha \in \omega_1$. Therefore there must be some $\alpha \in \omega_1$ such that $U \times Y$ meets $F \cap (X \times B_\alpha)$ in an infinite set. Since $X \times W$ will contain a cofinite subset of $F \cap (X \times B_\alpha)$, we then have that $U \times W$ meets $F \cap (X \times B_\alpha)$ (and hence F) in an infinite set.

So finally we complete the proof by showing that (in the extension by P) the family $\mathfrak{X}$ is not a countable union of 1-homogeneous sets. To see this, first we fix a P-name $\dot{\mathfrak{X}}$, for $\mathfrak{X}$. Suppose we have a P-name of such a sequence $\langle\dot{\mathfrak{X}}_n\rangle_n$ and a condition $p_0 \in P$ such that, $p_0 \Vdash \bigcup_n \dot{\mathfrak{X}}_n = \dot{\mathfrak{X}}$, and for each n, $p_0 \Vdash [\dot{\mathfrak{X}}_n]^2 \subset K_1$.

For better readability, let $A\backslash m$ abbreviate $A\backslash\{x_i : i < m\}$ for $A \subset X$ and $m \in \omega$, and similarly let $B\backslash m$ abbreviate $B\backslash\{y_j : j < m\}$ for $B \subset Y$. Recall that we showed that, for each $(A, B) \in \mathcal{A}_{x_0} \times \mathcal{B}_{y_0}$, there exists m such that $((A\times Y)\cap(X\times B))\cap E_m = \emptyset$. Therefore it follows that there is a sufficiently large m such that p_0 forces that $(\widetilde{A \setminus m}, \widetilde{B \setminus m})$ is a member of $\dot{\mathfrak{X}}$. Furthermore, there is an n and a $p < p_0$ in P, such that $p \Vdash (\widetilde{A \setminus m}, \widetilde{B \setminus m}) \in \dot{\mathfrak{X}}_n$. Let us define

$$\mathfrak{X}_{p,n,m} = \{(A, B) \in \mathcal{A}_{x_0} \times \mathcal{B}_{y_0} : p \Vdash (\widetilde{A \setminus m}, \widetilde{B \setminus m}) \in \dot{\mathfrak{X}}_n\}.$$

It is obvious that $\bigcup\{\mathfrak{X}_{p,n,m} : p \in P, n, m \in \omega\}$ should equal $\mathcal{A}_{x_0} \times \mathcal{B}_{y_0}$. We will prove our claim by proving that this is not the case. First let us enumerate $P \times \omega \times \omega$ in order type ω as $\{(p_k, n_k, m_k) : k \in \omega\}$ and we will construct, by induction on k, a descending sequence $\{X_k \times Y_k : k \in \omega\}$ of subspaces of $X \times Y$ (with $X_0 = X$ and $Y_0 = Y$). To guide this induction we fix an ultrafilter $\mathcal{W}$ on $\omega \times \omega$ which is not a P-filter. We also choose a sequence $\{a_j : j \in \omega\}$ converging to x_0 and $\{b_l : l \in \omega\}$ a sequence converging to y_0. At any stage k in the induction we will let (p, n, m) denote the triple (p_k, n_k, m_k) and we deal with $\mathfrak{X}_{p,n,m}$. For each k, let $\mathbb{A}_k = \{A\backslash m : \exists B \;\; (A, B) \in \mathfrak{X}_{p,n,m}\}$ and $\mathbb{B}_k = \{B\backslash m : \exists A \;\; (A, B) \in \mathfrak{X}_{p,n,m}\}$ for $n \in \omega$. As an induction hypothesis we will assume that, for all m, $\{(j, l) : (a_j, b_l) \in [E_m \cap (X_k \times Y_k)]'\} \in \mathcal{W}$. This is true for X_0 and Y_0 as E_m is a dense set of $X \times Y$ for all $m \in \omega$. The construction of X_{k+1} and Y_{k+1} will also ensure that, for each pair $(A, B) \in \mathfrak{X}_{p_k,n_k,m_k}$, one of $A \cap X_{k+1}$ and $B \cap Y_{k+1}$ will be finite.

Now we show the inductive step. Let $S_k = \bigcup \mathbb{A}_k$ and $T_k = \bigcup \mathbb{B}_k$. Now a key step in the proof is that since $p_0 \Vdash [\dot{\mathfrak{X}}_n]^2 \subset K_1$, there must exist $\overline{m}$ such that $(S_k \times T_k) \cap E_{\overline{m}} = \emptyset$. In fact choose $\bar{m}$ larger than each of m and $dom(p)$ and assume that $(x, y) \in (S_k \times T_k) \cap E_{\overline{m}}$ is not empty. Extend p to some $\bar{p}$ so that $\bar{p}(\bar{m}) = (x, y)$ and observe that $\bar{p} \Vdash (x, y) \in \dot{F}$. Since $(x, y) \in S_k \times T_k$ there are (A_0, B_0) and (A_1, B_1) in $\mathfrak{X}_{p,n,m}$ such that $x \in A_0 \setminus m \in \mathbb{A}_k$ and $y \in B_1 \setminus m \in \mathbb{B}_k$ so that $\bar{p} \Vdash (x, y) \in \dot{F} \cap ((A_0 \setminus m) \times (B_1 \setminus m))$. However notice also that $(x, y) \in (\widetilde{A_0 \setminus m} \cap \widetilde{B_1 \setminus m})$ and so $\bar{p} \Vdash \langle(\widetilde{A_0 \setminus m}, \widetilde{B_0 \setminus m}), (\widetilde{A_1 \setminus m}, \widetilde{B_1 \setminus m})\rangle \in K_0$. Of course this contradicts that p forces that this pair is in K_1.

Now we are ready to define $X_{k+1} \subset X_k$ and $Y_{k+1} \subset Y_k$. If for all $\bar{m} > m$,

$$\{(j, l) : (a_j, b_l) \in [E_{\bar{m}} \cap ((X_k \setminus S_k) \times Y_k)]'\} \in \mathcal{W}$$

then put $X_{k+1} = X_k \setminus S_k$ and $Y_{k+1} = Y_k$. Otherwise we set $X_{k+1} = X_k$ and $Y_{k+1} = Y_k \setminus T_k$. To show that this works we must show that for all $\bar{m} > m$,

$$\{(j, l) : (a_j, b_l) \in [E_{\bar{m}} \cap (X_k \times (Y_k \setminus T_k))]'\} \in \mathcal{W}\ .$$

If this fails, then there is an $\bar{m} > m$ such that

$$\{(j, l) : (a_j, b_l) \notin [E_{\bar{m}} \cap ((X_k \setminus S_k) \times Y_k)]' \cup [E_{\bar{m}} \cap (X_k \times (Y_k \setminus T_k))]'\} \in \mathcal{W}\ .$$

However this implies that $\{(j, l) : (a_j, b_l) \in \overline{E_{\bar{m}} \cap (S_k \times T_k)}\}$ is a member of $\mathcal{W}$, which is impossible since it contradicts the fact that $S_k \times T_k$ is disjoint from E_m.

So we select all the X_k's and Y_k's satisfying our induction hypothesis. According to our construction, for each k, there is $j_k > k$ such that the sequence a_{j_k} is in

X'_k. Now is the place where we use the hypothesis that X is Fréchet. For each k, choose a sequence $J_k \subset X_k$ converging to a_{j_k}. Since the sequence $\{a_{j_k}\}_k$ converges to x_0, we have that x_0 is in the closure of $\bigcup_k J_k$. Therefore there is a sequence $A \subset \bigcup_k J_k$ converging to x_0. By construction we have that $A \setminus X_k$ is finite for all k. By the similar argument as above we get a sequence B converging to y_0 with the property that $B \setminus Y_k$ is finite for all k. Therefore $(A, B) \in \mathcal{A}_{x_0} \times \mathcal{B}_{y_0}$ but clearly $(A, B) \notin \bigcup \mathfrak{X}_{p,n,m}$. □

4. Open Problems

We now know that SS$^+$ is productive for countable spaces. So the very natural question would be, is that true in general? That is,

Problem 3.1 Suppose X and Y are two SS$^+$ spaces. Must $X \times Y$ be SS$^+$?

Problem 3.2 Is the product of an SS space with an SS$^+$ space always SS?

We recall a problem from [1]. It is shown in [1] that if $\kappa < \mathfrak{d}$ and if $\kappa = \mathfrak{c}$ ([4]), then 2^κ has a dense SS subspace, so what happens in general?

Problem 3.3 Is there a $\kappa < \mathfrak{c}$ such that 2^κ fails to have a dense SS subspace?

Problem 3.4 Is there a ZFC example of a countable space which is Fréchet but not SS$^+$?

Gruenhage's question about Markov strategies for SS$^+$ spaces remains open as does the general problem about the product of Fréchet space.

Problem 3.5 Does the property of being SS$^+$ imply Markov SS ?

Problem 3.6 Is it consistent that the product of two separable Fréchet spaces is SS?

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