MORE ABOUT SPACES WITH A SMALL DIAGONAL

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Abstract. Hušek defines a space X to have a *small diagonal* if each uncountable subset of X^2 disjoint from the diagonal, has an uncountable subset whose closure is disjoint from the diagonal. Hušek proved that a compact space of weight ω_1 which has a small diagonal, will be metrizable, but it remains an open problem to determine if the weight restriction is necessary. It has been shown to be consistent that each compact space with a small diagonal is metrizable, in particular, Juhasz and Szentmiklossy [JS92] proved that this holds in models of CH. In this paper we prove that it also follows from the Proper Forcing Axiom (PFA). We also present two (consistent) examples of countably compact non-metrizable spaces with small diagonal, one of which maps perfectly onto ω_1.

1. Introduction

We refer the reader to Gruenhage's interesting article [Gru02] for more background on spaces with small diagonal. In particular, as mentioned there, H.X. Zhou [Zho82] is responsible for broadening the question to countably compact and Lindelöf spaces, while Hušek originally asked about compact and ω_1-compact spaces. As mentioned in the abstract, we prove that PFA implies that compact spaces with small diagonal are metrizable. This is the content of the second section. In the third section, we present two constructions of countably compact spaces with small diagonals. The first, from the hypothesis $\diamondsuit^+$, maps perfectly onto ω_1 with metric fibers. The second example is presented because the set-theoretic hypothesis that we are able to use is quite weak.

In this section we review some of the already established results concerning spaces with small diagonal that will be useful in our proofs. It is easily seen that a space with a G_δ-diagonal has a small diagonal. The Sorgenfrey line is a well-known example of a Lindelöf space with a G_δ-diagonal which is not metrizable. However for countably compact spaces the following very interesting result is well known.

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Proposition 1. [Cha76] *A countably compact space with a G_δ-diagonal is metrizable.*

Proposition 2. [JS92] *A compact space with a small diagonal will have countable tightness.*

Proposition 3. [Dow88b] *A countably compact space is metrizable if each of its subspaces of cardinality at most $\aleph_1$ is metrizable.*

Proposition 4. [Huš77] *A compact space of weight at most $\aleph_1$ is metrizable if it has a small diagonal.*

Corollary 5. *If a compact space has a small diagonal, then it is metrizable if each of its separable subspaces is metrizable.*

Proof. By Proposition 3, we may assume that X has a dense subset of cardinality at most $\aleph_1$ and, by 2, that X has countable tightness. Therefore X can be written as an increasing union of compact separable subspaces. If each of these is metrizable, then each has countable weight. In addition, X would then have a net of cardinality $\aleph_1$. Since the weight of a compact space is equal to the minimum cardinaity of a net (i.e. weight is equal to net weight), we would have that X has weight at most ω_1, and so by Proposition 4, X is metrizable. □

Proposition 6. [Gru02] *A first-countable hereditarily Lindelöf space X with a small diagonal, will have a G_δ-diagonal.*

2. PFA and compact spaces with small diagonal

As is well-known, it follows from PFA that there is no S-space.

Proposition 7. [Tod89] *PFA implies that each hereditarily separable (hS) space is also hereditarily Lindelöf (hL).*

Proposition 8. [Tod89] *PFA implies OCA: if X is a separable metric space and K_0 is a symmetric open subset of $X^2 \setminus \Delta_X$, then either there is an uncountable $Y \subset X$ such that $Y^2 \setminus \Delta_X \subset K_0$, or, X can be covered by a countable family $\{X_n : n \in \omega\}$ such that for each n, X_n^2 is disjoint from K_0.*

Our main result of this section is the following theorem.

Theorem 9. *PFA implies that each compact space with a small diagonal is metrizable.*

By Corollary 5, we may assume that our space of interest is separable and we will use the following characterization from [Gru02, 1.2].

Proposition 10. *A Lindelöf space X has a small diagonal iff for each uncountable family of pairs of distinct points of X, $\{(x_\alpha, y_\alpha) : \alpha \in \omega_1\}$, there is an uncountable $A \subset \omega_1$ such that $\{x_\alpha : \alpha \in A\}$ and $\{y_\alpha : \alpha \in A\}$ are separated by disjoint open F_σ-sets.*

The following result is established by proving that f is a closed mapping on Y, hence a homeomorphism.

Proposition 11. *If X is countably compact and f is a continuous function from X into a metric space and $Y \subset X$ is such that f is one-to-one on Y (i.e. $f^{-1}(f(y)) = \{y\}$ for each $y \in Y$), then Y is metrizable.*

In case it may have independent interest, we prove the following strengthening of Theorem 9.

Lemma 12. *PFA implies that each separable non-metrizable sequentially compact space X will contain a family of pairs $\{(y_\alpha, z_\alpha) : \alpha \in \omega_1\}$ such that for each uncountable $A \subset \omega_1$, $\{y_\alpha : \alpha \in A\}$ and $\{z_\alpha : \alpha \in A\}$ are not separated by disjoint open sets.*

Proof. We may assume that ω is a dense subset of our space X and that X is embedded in $[0,1]^\kappa$ for some cardinal κ.

We know by Proposition 3 that X must contain a non-metrizable subspace of cardinality ω_1, but we will choose a special subspace.

If X is hL, then it would be compact and first countable, hence by Proposition 6, X would be metrizable. Therefore X is not hL so it has a right-separated subspace $\{x_\alpha : \alpha \in \omega_1\}$ (i.e. for each $\alpha < \omega_1$, the set $\{x_\beta : \beta < \alpha\}$ is relatively open. Additionally, since we are assuming PFA, by Proposition 7 the subspace $\{x_\alpha : \alpha < \omega_1\}$ can not be hS, hence it has an uncountable discrete subspace. Therefore we may assume that $\{x_\alpha : \alpha \in \omega_1\}$ is a discrete subspace of X. For each $\alpha < \omega_1$, let W_α, U_α be open subsets of X so that $x_\alpha \in W_\alpha \subset \overline{W_\alpha} \subset U_\alpha$, and $\overline{U_\alpha} \cap \{x_\beta : \alpha \neq \beta < \omega_1\}$ is empty.

For each $\alpha \neq \gamma$, $x_\alpha \notin \overline{U_\gamma}$, hence the family $\{\omega \cap W_\alpha \setminus \overline{U_\gamma} : \gamma \neq \alpha\}$ has the finite intersection property. Since PFA implies MA(ω_1), there is an infinite set $a_\alpha \subset W_\alpha \cap \omega$ such that $a_\alpha \cap \overline{U_\gamma}$ is finite for all $\gamma \neq \alpha$. Since X is sequentially compact, we may assume that there is a $y_\alpha \in \overline{W_\alpha}$ such that a_α converges to y_α. Furthermore, by enlarging W_α by a small amount, we may assume that $y_\alpha \in W_\alpha \subset \overline{W_\alpha} \subset U_\alpha$.

Now we have chosen a set $Y = \{y_\alpha : \alpha \in \omega_1\}$ so that $\omega \cup Y$ is not metrizable, and a family of sequences $a_\alpha \subset \omega$ such that a_α converges to y_α. We are ready to apply OCA. Let $\mathcal{X}$ denote the family of all pairs of subsets (a, b) of ω such that there is an $\alpha \in \omega_1$ and a $z \in X \setminus Y$ such that $a = a_\alpha$, $a \cap b$ is empty, and b converges to z.

We topologize $\mathcal{X}$ by the usual separable metric topology where for each $n \in \omega$, $[(a,b)_n]$ denotes the family of all pairs $(a',b') \in \mathcal{X}$ such that $a' \cap n = a \cap n$ and $b' \cap n = b \cap n$.

Define a set $K_0 \subset \mathcal{X}^2$ by $(\ (a_\alpha, b_\alpha),\ (a_\beta, b_\beta)\ ) \in K_0$ if $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha)$ is not empty. Notice that it follows that $\alpha \neq \beta$ since otherwise we'd have $(a_\alpha \cup a_\beta) \cap (b_\alpha \cup b_\beta)$ is empty.

One can see that K_0 is open by simply choosing n large enough so that $a_\alpha \cap n \neq a_\beta \cap n$ and $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha)$ has a point in it below n.

Assume that $A \subset \omega_1$ is uncountable and $\{(a_\alpha, b_\alpha) : \alpha \in A\}$ is such that

$$(\ (a_\alpha, b_\alpha),\ (a_\beta, b_\beta)\ ) \in K_0 \ \text{ for each } \alpha \neq \beta \in A\ .$$

For each $\alpha \in A$, let $z_\alpha \in X$ be the point such that b_α converges to z_α.

Claim 1. The family $\{(y_\alpha, z_\alpha) : \alpha \in \omega_1\}$ satisfies the conclusion of the Lemma.

Assume that U, V are disjoint open subsets of X such that $y_\alpha \in U$ and $z_\alpha \in V$ for all $\alpha \in A'$ for some uncountable $A' \subset A$. By shrinking A' we may assume that there is an $n \in \omega$ such that $a_\alpha \setminus U \subset n$ and $b_\alpha \setminus V \subset n$ for all $\alpha \in A'$. Furthermore, we may assume that $a_\alpha \cap n$ and $b_\alpha \cap n$ are independent of $\alpha \in A'$. Clearly then, if α, β are distinct members of A' we have that $n \cap a_\alpha \cap b_\beta = n \cap a_\beta \cap b_\beta$ is empty, and $a_\alpha \setminus n \subset U$ while $b_\beta \setminus n \subset V$, contradicting that $(\ (a_\alpha, b_\alpha), (a_\beta, b_\beta)\ ) \in K_0$.

To finish the proof, we assume that $\mathcal{X}_n$ are subsets of $\mathcal{X}$ with the property that $\mathcal{X}_n^2$ is disjoint from K_0 and show that $\mathcal{X}$ is not equal to $\bigcup_n \mathcal{X}_n$. For each n, let $Y_n = \bigcup\{a_\alpha :\ (\exists b)\ (a_\alpha, b) \in \mathcal{X}_n\}$.

Let M be a countable elementary submodel of $H(\theta)$ for some large enough regular cardinal θ and assume that $\{y_\alpha : \alpha \in \omega_1\}$, $\{a_\alpha : \alpha \in \omega_1\}$, $\{\mathcal{X}_n : n \in \omega\}$, and the embedding of X as a subspace of $[0,1]^\kappa$ are each elements of M.

Let f denote the projection map from X into $[0,1]^{M\cap\kappa}$. Let $\delta = M \cap \omega_1$ (the minimum ordinal not in M). Let $\{U_m : m \in \omega\}$ be a descending family of basic open subsets of $[0,1]^{\kappa\cap M}$ (using rational intervals in finitely many cooordinates) which form a neighborhood base at $f(y_\delta)$. Although the family $\{U_m : m \in \omega\}$ is actually disjoint from M, the set $f^{-1}(U_m)$, for each m, is an open subset of X which is an element of M. Note that for $\beta \in M \cap \omega_1$, $f(y_\beta) \neq f(y_\delta)$ since $W_\beta \in M$ and $f(y_\gamma) \notin f(W_\beta)$ for all $\gamma > \beta$.

Let $J = \{n \in \omega : a_\delta \subset Y_n\}$.

Claim 2. There is an infinite set $b \subset \omega$ converging to a point $z' \in X \setminus Y$ such that for each $m \in \omega$,

$$b \setminus \bigcap_{n \in J \cap m} Y_n \text{ is finite .}$$

To find our set b we again use $\mathrm{MA}(\omega_1)$ to assert that there is a subset $b \subset \omega \setminus a_\delta$ such that b is almost contained in $\bigcap_{n \in J \cap m} Y_n \cap f^{-1}(U_m) \setminus (W_\delta \cup W_\gamma)$ for each $\gamma > \delta$.

To see this it is sufficient to show that

$$\bigcap_{n \in J \cap m} Y_n \cap f^{-1}(U_m) \setminus (\overline{W_{\gamma_0}} \cup \cdots \cup \overline{W_{\gamma_\ell}})$$

is infinite for each $\ell \in \omega$ and $\delta \leq \gamma_0 < \cdots < \gamma_\ell < \omega_1$.

This again follows from the fact that a_δ is almost contained in the set $A = \bigcap_{n \in J \cap m} Y_n \cap f^{-1}(U_m)$ which is in M. By elementarity, there is a $\beta \in M$ such that a_β is almost contained in A. Since such an a_β is almost disjoint from each W_γ with $\gamma \geq \delta$, we have shown that the above set is infinite.

We may shrink b so that it converges to some $z \in X$ and we observe that $f(z) = f(y_\delta)$. We check that $z \notin Y$. Since b is almost disjoint from W_γ for all $\gamma \geq \delta$, we have that $z \neq y_\gamma$ for all $\gamma \geq \delta$. On the other hand, our construction has ensured that $f(y_\beta) \neq f(y_\delta)$ for each $\beta \in M$.

Finally then, we have produced a pair $(a_\delta, b) \in \mathcal{X}$. Assume that $(a_\delta, b) \in \mathcal{X}_n$, hence $a_\delta \subset Y_n$. By construction, it follows that $n \in J$, hence b is almost contained in Y_n. Choose any $j \in b \cap Y_n$ and fix $(a_\beta, b_\beta) \in \mathcal{X}_n$ such that $j \in a_\beta$. Since $b \cap a_\delta$ is empty, it follows that $\beta \neq \delta$. However, we have that $(a_\beta \cap b) \cup (a_\delta \cap b_\beta)$ is not empty, which means that the pair $(\ (a_\delta, b),\ (a_\beta, b_\beta)\ )$ is in K_0, a contradiction. □

3. Countably Compact with small diagonal

Gruenhage[Gru02] has shown that it is consistent with CH that there are no countably compact non-metrizable space with a small diagonal [Gru02, 4.2]. He has also shown that it is consistent with the failure of CH that there is a countably compact non-metrizable space with a small diagonal [Gru02, 4.3]. In this section we generalize these results in two ways. In the first result we prove that it is consistent with CH that there is a perfect first-countable preimage of ω_1 (with metrizable fibers) which has a small diagonal (Theorem 13). This example answers several of the questions in [Gru02]. Secondly, we establish from a much weaker set-theoretic hypothesis than that in [Gru02, 4.3], that there is a countably compact non-metrizable space with a small diagonal (Theorem 16).

Recall that the classical $\diamondsuit^+$ (see [1]) is the following statement: *There are sets $\mathcal{A}_\alpha \subset \mathcal{P}(\alpha)$ for $\alpha < \omega_1$, such that each $|\mathcal{A}_\alpha| \leq \omega$ and for each $A \subset \omega_1$, there is a club $F \subset \omega_1$, such that for every $\alpha \in F$, $A \cap \alpha \in \mathcal{A}_\alpha$ and $F \cap \alpha \in \mathcal{A}_\alpha$.*

Theorem 13 ($\diamondsuit^+$)**.** *There is a perfect preimage of ω_1 with a small diagonal.*

Proof. Let $C = \{0,1\}^\omega$ be a copy of a Cantor set. Let $X = \omega_1 \times C$. π_1 and π_2 denote projections of X onto ω_1 and C respectively. For every $\alpha \in \omega_1$, $C_\alpha, X_{<\alpha}$, and $X_{\leq\alpha}$ are subsets of X, where $C_\alpha = \{\alpha\} \times C$, $X_{<\alpha} = \alpha \times C$, and $X_{\leq\alpha} = (\alpha + 1) \times C$. We put $X_{<\omega_1} = X$. ϕ_1 and ϕ_2 are projections of X^2 onto corresponding coordinates.

We define a topology on X in such a way that π_1 is a continuous perfect map. We define topology at points of C_α recursively for $\alpha \in \omega_1$. Each $X_{\leq\alpha}$ will be a subspace and an open subset of X, so a typical neighborhood of $x \in C_\alpha$ in $X_{\leq\alpha}$ will be a typical neighborhood of this x in X.

We introduce some notation for the properties that will ensure that X will have a small diagonal. Let $A = \{a_\alpha \in X^2 : \alpha < \beta\}$ for some $\beta \leq \omega_1$. We say that A is *simple* if $\pi_1(\phi_1(A)) \cup \pi_1(\phi_2(A))$ is discrete and for every $\alpha' < \alpha'' < \beta$ and for every combination of indexes $i, j \in \{1, 2\}$, $\pi_1(\phi_i(a_{\alpha'})) < \pi_1(\phi_j(a_{\alpha''}))$ A simple set A is well-ordered in the sense that for every $B \subset A$, there is $b \in B$ with the smallest (among elements of B) first and second coordinates.

In what follows, we use the following version of $\diamondsuit^+$: *There are sequences $\mathcal{A} = \{A^m_\alpha \subset (X_{<\alpha})^2 : \alpha \in \omega_1, m \in \omega\}$, $\mathcal{B} = \{B^n_\alpha \subset \alpha : \alpha \in \omega_1, n \in \omega\}$ such that for every simple $A \subset X^2$ there are a club $F \subset \omega_1$ and sequences $\{m(\alpha) \in \omega : \alpha \in \omega_1\}$, $\{n(\alpha) \in \omega : \alpha \in \omega_1\}$ such that for every $\alpha \in F$, $A \cap (X_{<\alpha})^2 = A^{m(\alpha)}_\alpha$ and $F \cap \alpha = B^{n(\alpha)}_\alpha$.*

The topology on X will satisfy the following conditions for every $\alpha \in \omega_1$:

(1_α) C_α is homeomorphic to a Cantor set.

(2_α) $Ind(X_{<\alpha}) = 0$.

(3_α) $X_{\leq\alpha}$ is a compact space.

(4_α) $\pi_{1\restriction X_{\leq\alpha}}$ is a continuous map.

(5_α) If α is a successor ordinal, then C_α is a clopen in $X_{\leq\alpha}$.

Let $\beta \leq \omega_1$ and let $A = \{a_\alpha \in X^2 : \alpha < \beta\}$ be a subset of X^2. We say that A is *thin* if $\overline{\phi_1(A)} \cap \overline{\phi_2(A)} = \emptyset$. For every $\beta \in \omega_1$, A is *simple in* β (respectively, A is *thin in* β) if $A \cap (X_{<\beta})^2$ is simple (respectively, thin) in the space $X_{<\beta}$. For every simple $A \subset X^2$, we denote by $lim(A)$ the set $\{\pi_1(\phi_1(x)) : x$ is a limit point of A in $X^2\}$ ($=\{\pi_1(\phi_2(x)) : x$ is a limit point of A in $X^2\}$). Finally, let $A \subset X^2$ be simple and let $F \subset \gamma$ be a nonempty closed set. When the context is clear, for each $\delta \in F$, δ^+ will denote the next smallest element of $F \cup \{\gamma\}$, i.e. δ^+ is γ if $\delta = \max(F)$. If $A \cap (X_{<\delta^+} \setminus X_{<\delta})^2$ is a nonempty set, denote by a_δ the smallest element of this set. If $A \cap (X_{<\delta^+} \setminus X_{<\delta})^2$ is empty, we put $a_\delta = \emptyset$. Denote $\{a_\delta : \delta \in F\}$ by $A_{\restriction F}$. If F is empty, let $A_{\restriction F}$ be empty too.

Let ($*_\alpha$) denote the following condition on a pair $(m, n) \in \omega^2$: A^m_α is simple in α, B^n_α is closed in α, and $A^m_{\alpha\,\restriction B^n_\alpha \cap lim(A^m_\alpha)}$ is thin in α.

The following condition will ensure that X has a small diagonal:

(6_α) If (m, n) satisfies ($*_\alpha$), then $A^m_{\alpha\,\restriction B^n_\alpha \cap lim(A^m_\alpha)}$ is thin in $\alpha + 1$.

Note that if (m, n) satisfies ($*_\gamma$), then condition (6_γ) essentially means that γ is *not* the minimal ordinal such that

$$\overline{\phi_1(A^m_{\gamma\,\restriction B^n_\gamma \cap lim(A^m_\gamma)})} \cap \overline{\phi_2(A^m_{\gamma\,\restriction B^n_\gamma \cap lim(A^m_\gamma)})} \cap C_\gamma \neq \emptyset \ .$$

Using (6_α), we will show that for every uncountable simple subset A of X^2, $\overline{\phi_1(A_{\restriction G})} \cap \overline{\phi_2(A_{\restriction G})} = \emptyset$ since there is no minimal $\gamma \in \omega_1$ such that $\overline{\phi_1(A_{\restriction G})} \cap \overline{\phi_2(A_{\restriction G})} \cap C_\gamma \neq \emptyset$ (here G is a certain club in ω_1 which depends on A and which is produced with the help of $\diamondsuit^+$-sequences).

Now let $\gamma \in \omega_1$ and suppose that topology has been defined on $X_{<\gamma}$ and satisfies (1_α) – (6_α) for every $\alpha < \gamma$.

Case 1. γ is a successor ordinal.

In this case, let C_γ be a clopen subset of $X_{\leq\gamma}$ which is homeomorphic to a Cantor set. Then (1_γ) – (6_γ) are obviously satisfied.

Case 2. γ is a limit ordinal.

In this case, $X_{<\gamma}$ is a free topological sum of sets with zero large inductive dimension, so (2_γ) holds. Enumerate all pairs $(m, n) \in \omega^2$ which satisfy ($*_\gamma$) by $\{(m_k, n_k) : k \in \omega\}$. We can assume that the family of such pairs is infinite (if not, replace some A^m_γ, B^n_γ so that infinitely many pairs $(m, n) \in \omega^2$ satisfy ($*_\gamma$)). For every $k \in \omega$, let $\mathcal{I}_k = \{0, 1\}^k$ be the set of all binary sequences of the length k, and let $\mathcal{I} = \cup\{\mathcal{I}_k : k \in \omega\}$. Let U_0, U_1 be complementary clopen subsets of $X_{<\gamma}$ such that $\phi_1(A^{m_0}_{\gamma\,\restriction B^{n_0}_\gamma \cap lim(A^{m_0}_\gamma)}) \subset U_0$ and $\phi_2(A^{m_0}_{\gamma\,\restriction B^{n_0}_\gamma \cap lim(A^{m_0}_\gamma)}) \subset U_1$. Such sets exist according to (2_γ) and ($*_\gamma$). Now assume that a family $\{U_I : I \in \mathcal{I}_k\}$ of clopen subsets of $X_{<\gamma}$ has been defined for some $k \in \omega$. For every $I \in \mathcal{I}_k$, let $U_{I^\frown 0}$, $U_{I^\frown 1}$ be complementary clopen subsets of U_I such that $\phi_1(A^{m_k}_\gamma{}_{\restriction B^{n_k}_\gamma \cap lim(A^{m_k}_\gamma)}) \cap U_I \subset U_{I^\frown 0}$ and $\phi_2(A^{m_k}_\gamma{}_{\restriction B^{n_k}_\gamma \cap lim(A^{m_k}_\gamma)}) \cap U_I \subset U_{I^\frown 1}$. Let $c = \{i_p \in \{0,1\} : p \in \omega\}$ be an element of C. Then a typical neighborhood of (γ, c) in $X_{\leq\gamma}$ is $(\{\gamma\} \times C_I) \cup (U_I \setminus X_{\leq\delta})$ for various $I \in \mathcal{I}$ and $\delta < \gamma$, where C_I denotes the set of all elements of C which extend I.

Now we check properties (1_γ) and (3_γ) – (6_γ) in Case 2.

It is well-known that $\{C_I : I \in \mathcal{I}\}$ is a base of C. Hence $\{\{\gamma\} \times C_I : I \in \mathcal{I}\}$ is a base of C_γ. Therefore C_γ is homeomorphic to a Cantor set and (1_γ) holds.

Assume towards contradiction that (3_γ) fails. Then there is an infinite subset S of $X_{\leq\gamma}$ without a complete accumulation point in $X_{\leq\gamma}$. Therefore for every $x \in C_\gamma$, there is $I(x) \in \mathcal{I}$ such that the sets $U_{I(x)} \cap S$ and $C_{I(x)} \cap S$ have smaller cardinality than S. The family$\{C_{I(x)} : x \in C_\gamma\}$ is an open cover of C_γ, so it contains a finite subcover which we denote by $\mathcal{U}$. It then follows from the definition of topology at C_γ that $\cup\mathcal{U}$ contains $X_{\leq\gamma} \setminus X_{\leq\delta}$ for some $\delta < \gamma$. Therefore $|S \setminus X_{\leq\delta}| < |S|$, and $S \cap X_{\leq\delta}$ is an infinite subset of $X_{\leq\delta}$ with no complete accumulation point in $X_{\leq\delta}$. However, $X_{\leq\delta}$ is a compact space according to (3_δ). This contradiction proves (3_γ).

(4_γ) is true since (4_α) is true for every $\alpha < \gamma$ and since $(\pi_1)^{-1}((\alpha, \gamma])$ is open in $X_{\leq\gamma}$ from the definition of topology on $X_{\leq\gamma}$.

(5_γ) hold trivially in Case 2, and

(6_γ) holds because $\bigcup\{U_{I^\frown 0} : I \in \mathcal{I}_k\}$ contains $\phi_1(A^{m_k}_\gamma{}_{\restriction B^{n_k}_\gamma \cap \lim(A^{m_k}_\gamma)})$ and, similarly, $\bigcup\{U_{I^\frown 1} : I \in \mathcal{I}_k\}$ contains $\phi_2(A^{m_k}_\gamma{}_{\restriction B^{n_k}_\gamma \cap \lim(A^{m_k}_\gamma)})$.

We next show that π_1 is a continuous perfect map. Fix $x \in X$, then there is $\alpha \in \omega_1$ such that $x \in X_{<\alpha}$. $X_{<\alpha}$ is an open neighborhood of x and $\pi_{1\restriction X_{<\alpha}}$ is a continuous map by (4_α). Therefore π_1 is continuous at x. Hence π_1 is a continuous map. Now fix a closed set $F \subset X$. To prove that $\pi_1(F)$ is closed in ω_1, it is enough to show that $\pi_1(F) \cap [0, \alpha]$ is closed for every $\alpha \in \omega_1$. The latter statement is true since $\pi_1(F) \cap [0, \alpha] = \pi_1(F \cap X_{\leq\alpha})$ and since $F \cap X_{\leq\alpha}$ is a compact set by (3_α). Further, every fiber of π_1 is a compact set by (1_α). So π_1 is a continuous perfect map.

The last step is to show that X has a small diagonal. Recall that $\Delta(X)$ denotes a diagonal of X. Fix an uncountable set $A \subset X^2$. If $A \cap (X \setminus X_{\leq\delta})^2$ is countable for some $\delta \in \omega_1$, then either $A \cap (X_{\leq\delta} \times (X \setminus X_{\leq\delta}))$ is uncountable, or $A \cap ((X \setminus X_{\leq\delta}) \times X_{\leq\delta})$ is uncountable, or $A \cap (X_{\leq\delta})^2$ is uncountable. In the first two cases, $A \setminus ((X_{\leq\delta})^2 \cup (X \setminus X_{\leq\delta})^2)$ is uncountable. In the third case, there is a neighborhood $U \subset (X_{\leq\delta})^2$ of $\Delta(X_{\leq\delta})$ such that $(A \cap (X_{\leq\delta})^2) \setminus U$ is uncountable. Either way, $U \cup (X \setminus X_{\leq\delta})^2$ is a neighborhood of $\Delta(X)$ which misses uncountably many elements of A. So we can assume that $A \cap (X \setminus X_{\leq\delta})^2$ is an uncountable set for every $\delta \in \omega_1$. Then A contains an uncountable simple set. Therefore, we can just assume that A is simple itself.

According to our version of $\diamondsuit^+$, there is a club $F \subset \omega_1$ and sequences $\{m(\alpha) \in \omega : \alpha \in \omega_1\}$, $\{n(\alpha) \in \omega : \alpha \in \omega_1\}$ such that $A \cap (X_{<\alpha})^2 = A_\alpha^{m(\alpha)}$ and $F \cap \alpha = B_\alpha^{n(\alpha)}$ for every $\alpha \in F$. The set $G = F \cap lim(A)$ is a club since $lim(A)$ is a club. We show that $A_{\restriction G}$ is an uncountable thin set. In particular, $\overline{A_{\restriction G}} \cap \Delta(X) = \emptyset$, which implies that X has a small diagonal. First of all, for every $\delta_1, \delta_2 \in G$ with $\delta_1 < \delta_2$, $(X_{<\delta_2} \setminus X_{<\delta_1})^2 \cap A$ is a nonempty set since A is simple and since $\delta_1, \delta_2 \in lim(A)$. Therefore $A_{\restriction G}$ is uncountable. Assume towards contradiction that $A_{\restriction G}$ is not thin. Then there is a minimal ordinal $\gamma \in \omega_1$ such that

$$\overline{\phi_1(A_{\restriction G})} \cap \overline{\phi_2(A_{\restriction G})} \cap C_\gamma \neq \emptyset. \tag{1}$$

It is possible to check that $\gamma \in G$, in particular $\gamma \in F$. Therefore $A_\gamma^{m(\gamma)} = A \cap (X_{<\gamma})^2$ and $B_\gamma^{n(\gamma)} \cap \gamma = F \cap \gamma$. Also $lim(A \cap (X_{<\gamma})^2) = lim(A) \cap \gamma$. This means that

$$\begin{aligned} A_\gamma^{m(\gamma)}{}_{\restriction B_\gamma^{n(\gamma)} \cap \lim(A_\gamma^{m(\gamma)})} &= (A \cap (X_{<\gamma})^2)_{\restriction (F\cap\gamma)\cap(lim(A)\cap\gamma)} \\ &= (A \cap (X_{<\gamma})^2)_{\restriction F \cap lim(A) \cap \gamma} = \\ &= (A \cap (X_{<\gamma})^2)_{\restriction G\cap\gamma} = A_{\restriction G} \cap (X_{<\gamma})^2. \end{aligned}$$

The latter set is thin in γ by the minimality γ. So the pair $(m(\gamma), n(\gamma))$ satisfies $(*_\gamma)$ and $A_\gamma^{m(\gamma)}{}_{\restriction B_\gamma^{n(\gamma)} \cap \lim(A_\gamma^{m(\gamma)}))}$ is thin in $\gamma + 1$ by (6_γ), hence

$$\overline{\phi_1(A_{\restriction G}) \cap (X_{<\gamma})^2} \cap \overline{\phi_2(A_{\restriction G}) \cap (X_{<\gamma})^2} \cap C_\gamma = \emptyset. \tag{2}$$

However, (2) contradicts (1) since $\phi_i(A_{\restriction G}) \cap C_\gamma = \emptyset$ for $i \in \{1, 2\}$. Therefore $A_{\restriction G}$ is thin. □

For our next result, we need another set-theoretic principle.

Definition 14. The statement **MA**(Cohen) is the Martin's Axiom type assertion that if P is a poset for adding any number of Cohen reals and if $\mathcal{D}$ is a family of fewer than $\mathfrak{c}$ dense subsets of P, then there is a filter $G \subset P$ which meets each member of $\mathcal{D}$.

The actual consequence of **MA**(Cohen) that we need is the following.

Proposition 15 (**MA**(Cohen))**.** *If $\mathcal{A}$ is a family of fewer than $\mathfrak{c}$ countably infinite sets, then there is set $Y \subset \bigcup \mathcal{A}$ such that neither Y nor $\bigcup \mathcal{A} \setminus Y$ contains any of the members of $\mathcal{A}$.*

The proof is completely straightforward and will be omitted.

Theorem 16 (**MA**(Cohen)+ $2^{\omega_1} = 2^{\omega}$)**.** *Each of the spaces $2^{\mathfrak{c}}$ and $\beta\omega$ contain dense countably compact subsets of cardinality $\mathfrak{c}$ which have small diagonals.*

Proof. We will define a subspace $X = \{x_\alpha : \alpha \in \mathfrak{c}\}$ of $2^{\mathfrak{c}}$ by induction on $\alpha \in \mathfrak{c}$. At stage α, we will have chosen, for each $\beta < \alpha$, the values of $x_\beta \restriction \alpha$ for each $\beta < \alpha$. At stage α we will extend each $x_\beta \restriction \alpha$ by defining its value at α, i.e. $x_\beta(\alpha)$ will be defined to be either 0 or 1, and we will also choose $x_\alpha \restriction \alpha + 1$ to be some member of $2^{\alpha+1}$.

In order to ensure that X will be countably compact, let $\{A_\gamma : \gamma \in \mathfrak{c}\}$ be an enumeration of the countably infinite subsets of $\mathfrak{c}$ such that $A_n = \omega \setminus n$ for each $n \in \omega$, and $A_\gamma \subset \gamma$ for all $\gamma \geq \omega$. Our inductive hypothesis will include the condition, for $\gamma \leq \alpha$, that the point $x_\gamma \restriction \alpha$ is a limit point (in the space 2^α) of the set $\{x_\beta \restriction \alpha : \beta \in A_\gamma\}$. We will also preserve that $\{x_n \restriction \alpha : n \in \omega\}$ is dense in 2^α.

In order to ensure that X will have a small diagonal, let $\{h_\gamma : \gamma \in \mathfrak{c}\}$ enumerate all functions from ω_1 into $[\mathfrak{c}]^2$ (the two element subsets of $\mathfrak{c}$) so that for each γ, the range of h_γ is contained in $[\gamma]^2$. We will arrange that for each α, there is an uncountable $I \subset \omega_1$ such that for each $\zeta \in I$, $h_\alpha(\zeta) = \{\zeta_0 < \zeta_1\}$ and $x_{\zeta_0}(\alpha) = 0$ and $x_{\zeta_1}(\alpha) = 1$. Note then that if $U(\alpha, 0)$ is the subset of $2^{\mathfrak{c}}$ consisting of all points that have value 0 at α and $U(\alpha, 1) = 2^{\mathfrak{c}} \setminus U(\alpha, 0)$, then $(U(\alpha, 0) \times U(\alpha, 0)) \cup (U(\alpha, 1) \times U(\alpha, 1))$ is an open neighborhood of the diagonal of X in X^2 but contains none of the points in $\{(x_{\zeta_0}, x_{\zeta_1}) : \zeta \in I \text{ and } h_\alpha(\zeta) = \{\zeta_0, \zeta_1\}\}$.

We begin the induction by choosing $\{x_n \restriction \omega : n \in \omega\}$ to be any countable dense subset of 2^ω. At stage $\alpha \geq \omega$, we may assume that we have chosen $x_\gamma \restriction \alpha$ for all $\gamma \in \alpha$ and have, by induction, preserved that $x_\gamma \restriction \alpha$ is a limit point of $\{x_\beta \restriction \alpha : \beta \in A_\gamma\}$. We must first choose a set $J \subset \alpha$ and define $x_\gamma(\alpha) = 0$ for all $\gamma \in J$ and $x_\gamma(\alpha) = 1$ for all $\gamma \in \alpha \setminus J$. Then we simply choose a point $x_\alpha \restriction \alpha + 1$ which is a limit point of $\{x_\beta : \beta \in A_\alpha\}$.

Define $\mathcal{A}_\alpha$ to be the family of all countably infinite sets B of α such that there is a basic clopen subset W of 2^α and a $\gamma < \alpha$ such that B is a cofinite subset of $\{\xi \in D_\gamma : x_\xi \restriction \alpha \in W\}$. Clearly $\mathcal{A}_\alpha$ has cardinality at most $|\alpha|$. By **MA**(Cohen), there is an uncountable set I_0 of ω_1 such that $\{\zeta_0 : \zeta \in I_0\}$ contains no member of $\mathcal{A}_\alpha$. Similarly, there is an uncountable $I_1 \subset I_0$ such that $\{\zeta_1 : \zeta \in I_1\}$ contains no member of $\mathcal{A}_\alpha$. Let $I_2 = \{\zeta_0, \zeta_1 : \zeta \in I_1\}$ and $\mathcal{A}_1$ be the family $\{A \setminus I_2 : A \in \mathcal{A}_\alpha\}$

and, by **MA**(Cohen), choose a set $J \subset \alpha \setminus I_2$ so that neither J nor $\alpha \setminus J$ contains any member of $\mathcal{A}_1$. Finally, define $x_\gamma(\alpha) = 0$ for all $\gamma \in J \cup \{\zeta_0 : \zeta \in I_1\}$, and $x_\gamma(\alpha) = 1$ for other $\gamma \in \alpha$.

It is routine, by the choice of $\mathcal{A}_1$, to prove that the inductive hypothesis that $x_\gamma \restriction \alpha + 1$ is in the closure of $\{x_\beta \restriction \alpha + 1 : \beta \in A_\gamma\}$ for each $\gamma \leq \alpha$. We have also preserved that $\{x_n \restriction \alpha + 1 : n \in \omega\}$ is dense since $\mathcal{A}_\alpha$ will list (the indices of) all the basic clopen subsets of 2^α intersected with $\{x_n \restriction \alpha : n \in \omega\}$.

This completes the proof for the dense subset of $2^\mathfrak{c}$. One could make a minor modification of the proof in order to arrange that one particular countable discrete subset of X had closure in $2^\mathfrak{c}$ which was homeomorphic to $\beta\omega$, and then the closure of this subset in X would be our desired dense subset of $\beta\omega$. However, instead we can make one very minor additional requirement on our construction, namely that $x_\gamma \neq x_\alpha$ for all $\gamma \neq \alpha$. Then let f be the canonical map from $\beta\omega$ onto $2^\mathfrak{c}$ which sends n to x_n for each n. For each integer n, let y_n denote $n \in \beta\omega$. Recursively choose $y_\alpha \in \beta\omega$ (for $\alpha \in \mathfrak{c}$) so that $f(y_\alpha) = x_\alpha$ and y_α is a limit point of $\{y_\beta : \beta \in A_\alpha\}$. Now f will be a one-to-one map from $Y = \{y_\alpha : \alpha \in \mathfrak{c}\}$ onto X and Y is easily seen to be a countably compact dense subset of $\beta\omega$. Then Y has a small diagonal since the map onto X is one-to-one. □

It would be interesting to determine if simply the failure of CH implies that there is a countably compact non-metrizable space with a small diagonal. Gruenhage's example was also initially ω_1-compact; the above construction can be modified to also ensure that every set of cardinality ω_1 has a complete accumulation point by using the following stronger consequence of **MA**(Cohen).

Proposition 17 (**MA**(Cohen))**.** *If $\mathcal{A}$ is a family of fewer than $\mathfrak{c}$ infinite sets, then there is set $Y \subset \bigcup \mathcal{A}$ such that for all $A \in \mathcal{A}$, each of $A \cap Y$ and $A \setminus Y$ have the same cardinality as that of A.*

4. Lindelöf and small diagonal

In this section we present a simple observation about Lindelöf spaces with a small diagonal which generalizes a result by Zhou [Zho82] for compact spaces. We also show, in this section, that the failure of CH implies there is a Lindelöf space with a small diagonal which does not have a G_δ-diagonal.

Proposition 18. *If a space X is Lindelöf and every continuous image of X has a small diagonal, then X has a G_δ-diagonal and can be condensed onto a metrizable space.*

Proposition 18 is actually a corollary to the following.

Proposition 19. *For a Lindelöf space X either*

(1) *X has a continuous image with weight (precisely) ω_1 which does not have a small diagonal, or*
(2) *X can be mapped onto a metric space by a one-to-one map.*

Proof. We may assume that X is embedded in $[0,1]^\kappa$ for some cardinal κ. For each set $I \subset \kappa$, let X_I denote the image of X by the projection map $\pi_I : [0,1]^\kappa \to [0,1]^I$. If I is countable, then of course X_I is metrizable; hence if there is a countable $I \subset \kappa$ such that $\pi_I \restriction X$ is one-to-one then the second condition holds. Therefore we assume otherwise, and inductively choose an increasing sequence $\{I_\alpha : \alpha \in \omega_1\}$ of countable sets as follows. Let I_0 be any non-empty countable subset of κ, and for limit α let I_α be the union $\bigcup_{\beta<\alpha} I_\beta$. For each α, if π_{I_α} is not one-to-one, then let x_α, y_α be a pair of points of X such that $\pi_{I_\alpha}(x_\alpha) = \pi_{I_\alpha}(y_\alpha)$. Choose $I_{\alpha+1} \supset I_\alpha$ large enough so that $\pi_{I_{\alpha+1}}(x_\alpha) \neq \pi_{I_{\alpha+1}}(y_\alpha)$.

Let $I = I_{\omega_1}$ and consider the sequence $\{(x'_\alpha, y'_\alpha) : \alpha \in \omega_1\} \subset X_I^2$ where $x'_\alpha = \pi_I(x_\alpha)$ and $y'_\alpha = \pi_I(y_\alpha)$. To show that X_I^2 does not have a small diagonal, let A be any uncountable subset of ω_1. Now by the hypothesis that X_I is Lindelöf we may choose a point $x' \in X_I$ which is a complete accumulation point of the set $\{x'_\alpha : \alpha \in A\}$. Of course, if U is any basic open neighborhood of x' in X_I then of the uncountably many $\alpha \in A$ with $x'_\alpha \in U$, we will also have that $y'_\alpha \in U$ for all but countably many of them. Therefore the point (x', x') on the diagonal of X_I is a limit point of $\{(x'_\alpha, y'_\alpha) : \alpha \in A\}$. □

Gruenhage shows in [Gru02] that it is consistent with CH (and holds in Godel's constructible universe L) that there is a Lindelöf space with a small diagonal which does not have a G_δ-diagonal. We show that the failure of CH alone will also provide such an example.

Theorem 20. *If $\mathfrak{c} > \omega_1$, then there is a Lindelöf space which has a small diagonal but not a G_δ-diagonal.*

Proof. Recall that a set $B \subset [0,1]$ is a Bernstein set if B meets, but does not contain, each uncountable closed subset of $[0,1]$. It is well-known that there is a family of $\mathfrak{c}$ many pairwise disjoint Bernstein sets, so let $\{B_\alpha : \alpha \in \omega_2\}$ be any subfamily of such a family. Our space X will simply be a subspace of the product space $(\omega_2+1) \times [0,1]$ where $\omega_2 + 1$ is the compact order topology on the set of ordinals. For each $\alpha < \omega_2$, we choose $X \cap (\{\alpha\} \times [0,1])$ to be $\{\alpha\} \times B_\alpha$ and we take $\{\omega_2\} \times [0,1] \subset X$. Basic open sets for X will be taken to be of the form $X \cap ([\alpha, \beta] \times I)$ for $\alpha \leq \beta \leq \omega_2$, α not a limit ordinal, and I any (relatively) open interval in $[0, 1]$.

For each $\lambda \leq \omega_2$, we show, by induction on λ, that the subspace

$$X_\lambda = X \cap ([0, \lambda] \times [0, 1])$$

is Lindelöf.

If λ has countable cofinality, then $X_\lambda = \bigcup_{\mu<\lambda} X_\mu \ \cup (\{\lambda\} \times B_\lambda)$ is a countable union of Lindelöf subsets, hence is Lindelöf itself. To see that X_λ is Lindelöf when λ has uncountable cofinality, let $\mathcal{W}$ be a cover of X_λ by basic open sets. Since $\{\lambda\} \times B_\lambda$ (or $\{\omega_2\} \times [0, 1]$) is Lindelöf, there is a countable subcollection $\mathcal{W}'$ of $\mathcal{W}$ whose union covers $X_\lambda \cap (\{\lambda\} \times [0, 1])$. We may assume that $\mathcal{W}'$ is the family $\{([\alpha_n, \lambda] \times I_n) : n \in \omega\}$ where each I_n is an open subinterval of $[0, 1]$. Since each B_α is Bernstein, $[0, 1] \setminus \bigcup_n I_n$ is countable. In addition, the family $\{B_\alpha : \alpha < \lambda\}$ are pairwise disjoint (and λ has uncountable cofinality), hence there is a $\gamma < \lambda$ such that $\alpha_n < \gamma$ for each n, and $B_\beta \subset \bigcup_n I_n$ for each $\gamma < \beta \leq \lambda$. It follows that $X_\lambda \setminus X_\gamma$ is contained in $\bigcup \mathcal{W}'$. Since X_γ is Lindelöf (by inductive assumption), it follows that $\mathcal{W}$ has a countable subcover of X_λ. This shows that X_λ is Lindelöf.

A similar argument shows that X does not have a G_δ diagonal. Indeed, if $\{U_n : n \in \omega\}$ is a family of open sets in X^2 which contain Δ_X, then for each n, there is a countable (finite in fact), cover, $\mathcal{W}_n$, of $\{\omega_2\} \times [0, 1]$ consisting of basic open sets W with the property that $W \times W \subset U_n$ for each $W \in \mathcal{W}_n$. Letting $\mathcal{W}'$ be the collection of all the $\mathcal{W}_n$, choose $\gamma < \omega_2$ as in the above argument. Fix any β with $\gamma < \beta < \omega_2$, and notice that for each $x \in B_\beta$, the pair $p_x = ((\beta, x)\ ,\ (\omega_2, x)) \in X^2$ has the property that for each n, there is a $W \in \mathcal{W}_n$ such that $p_x \in W \times W \subset U_n$. This proves that Δ_x is not equal to $\bigcap_n U_n$.

Finally we prove that X has a small diagonal. Observe that the second coordinate projection map into $[0, 1]$ is one-to-one on X_λ for $\lambda < \omega_2$. Therefore for each such λ, X_λ is a clopen subset of X which has a G_δ-diagonal. Let A be an uncountable subset of $X^2 \setminus \Delta_X$. If there is any $\lambda < \omega_2$ such that $A \cap X_\lambda^2$ is uncountable, then clearly there is an uncountable $B \subset A$ whose closure is disjoint from Δ_X. Similarly it follows that A meets the square of $\{\omega_2\} \times [0, 1]$ in a countable set. Therefore we may assume that A has cardinality ω_1 and there is a $\lambda < \omega_2$ such that each point in A has one point from X_λ and one point from $\{\omega_2\} \times [0, 1]$. It follows then that $X_\lambda^2 \cup (X \setminus X_\lambda)^2$ is a neighborhood of Δ_X which is disjoint from A. □

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