PFA AND COMPLEMENTED SUBSPACES OF ℓ_∞/c_0

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Abstract. The Banach space ℓ_∞/c_0 is isomorphic to the linear space of continuous functions on $\mathbb{N}^*$ with the supremum norm, $C(\mathbb{N}^*)$. Similarly, the canonical representation of the ℓ_∞ sum of ℓ_∞/c_0 is the Banach space of continuous functions on the closure of any non-compact cozero subset of $\mathbb{N}^*$. It is important to determine if there is a continuous linear lifting of this Banach space to a complemented subset of $C(\mathbb{N}^*)$. We show that PFA implies there is no such lifting.

1. Introduction

Our paper is motivated by the question ([3, 6]) of whether or not $C(\mathbb{N}^*)$ is primary. A Banach space X is primary if whenever X is written as the sum $A \oplus B$ of complemented subspaces, then one of A, B is isomorphic to X. Negrepontis [8, Corollary 3.2] showed that CH implies that the closure Y of a non-compact cozero subset of $\mathbb{N}^*$ is a retract of $\mathbb{N}^*$, and, therefore, there is a norm bounded linear lifting of the Banach space $C(Y)$ to a complemented subset of $C(\mathbb{N}^*)$. Later, Drewnowski and Roberts [3] established that the existence of such a lifting implied that $C(\mathbb{N}^*)$ is primary. It is already known to be consistent that there is no such lifting; an even stronger result was shown to hold in the Cohen model in [1]. However there is still a good reason to investigate this question under the hypothesis of the proper forcing axiom. We still have no clear path to deciding if $C(\mathbb{N}^*)$ is primary in the Cohen model but Koszmider [9, p577] has identified a very compelling conjecture (as we choose to call it) that $C(\mathbb{N}^*)$ is not primary in certain forcing extensions of PFA. Establishing properties of $C(\mathbb{N}^*)$ in these extensions is very similar to working within PFA itself (see [14, 12, 2]). We present our work as progress towards confirming that conjecture. The paper [4] announced similar results and gave reference to a paper in preparation for details. But even now, a number of years later, the details of a proof have not appeared and there appear to be problems with the sketch described in [4, p306-307]; we say more on this in Remark 2.1 after establishing more notation. Our own proof takes quite a different approach. It is modelled on the methods developed in [5, 11].

2. PFA implies no lifting

Let $\{A_n : n \in \omega\}$ be a partition of $\mathbb{N}$ into infinite sets. Let Y be the open subset $\bigcup_n A_n^*$ of $\mathbb{N}^*$. Consider the subspace $E = \{f \in C(\mathbb{N}^*) : f[Y] = \{0\}\}$. It is well-known (see [9, p574]) that there is a continuous lifting for $C(\overline{Y})$ if and only if the subspace E is complemented in the Banach space $(C(\mathbb{N}^*), \|\cdot\|_\infty)$. We take as

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our definition of E being complemented that there is a projection P from $C(\mathbb{N}^*)$ to E (a bounded linear operator) satisfying that $P^2(f) = P(f) \in E$ for all $f \in C(\mathbb{N}^*)$. Of course the norm of P is defined as the supremum of $\{\|P(f)\|_\infty : \|f\|_\infty = 1\}$.

This then provides a complement to E and an operator T defined by $T(f) = f - P(f)$ for $f \in C(\mathbb{N}^*)$ onto that complement. Again, it follows that T is bounded, linear, and satisfies that $T^2(f) = T(f)$. We may view T as a lifting of the functions from $C(\overline{Y})$ into $C(\mathbb{N}^*)$ since it follows that $T(f) \restriction \overline{Y} = f \restriction \overline{Y}$ for all $f \in C(\mathbb{N}^*)$. More precisely, for any $h \in C(\overline{Y})$, define $H(h)$ to be $T(f)$ where f is any $f \in C(\mathbb{N}^*)$ such that $h \subset f$. Then H is a continuous linear embedding (in fact, lifting) of $C(\overline{Y})$ into $C(\mathbb{N}^*)$.

Theorem 1 (PFA). *If $\{A_n : n \in \omega\}$ is a partition of $\mathbb{N}$ into infinite sets, then the subspace $E = \{f \in C(\mathbb{N}^*) : f[\bigcup_n A_n^*] = \{0\}\}$ is not complemented. Equivalently, there is no operator T as described above.*

We assume PFA for the remainder and that T is an operator as described in the paragraph immediately preceding the statement of the theorem. Following standard Stone-Cech compactification notation, the set of bounded (continuous) functions on $\mathbb{N}$ is denoted as $C^*(\mathbb{N})$. We fix any lifting of T to all of $C^*(\mathbb{N})$ in the sense that for all bounded $f \in C(\mathbb{N})$, $T(f) \in C^*(\mathbb{N})$ is chosen so that $[T(f)]^*$ is equal to $T(f^*)$.

So, we note that, for all $f \in C^*(\mathbb{N})$, $(f - T(f)) \restriction A_n \to 0$ for all n. Additionally, for $f \in C^*(\mathbb{N})$, we note that $\|f^*\|_\infty = 0$ (i.e. $f^* \equiv 0$) is equivalent to f converging to 0 on $\mathbb{N}$. We will say that two real-valued functions on $\mathbb{N}$ asymptotically agree if their difference converges to 0. Also, when we refer to the norm of a member of $C^*(\mathbb{N})$ we mean the asymptotic norm or the norm of f^*.

The set $\{A_n : n \in \omega\}^\perp$ is the ideal of subsets of $\mathbb{N}$ which are almost disjoint from each A_n. Let $\mathcal{I}$ denote the larger (dense) ideal of sets that are almost disjoint from A_n for all but finitely many n. As usual, $\mathcal{I}^+$ is the collection of sets which are not in this ideal. Note that a set $a \subset \mathbb{N}$ is in $\mathcal{I}^+$ if and only if the set $J_a = \{j : |a \cap A_j| = \omega\}$ is infinite. Unless mentioned otherwise, we will assume that $a \cap A_n$ is empty for $n \notin J_a$. Let $\mathcal{J} \subset \mathcal{I}^+$ denote the collection of those $a \in \mathcal{I}^+$ with the property that $J_{\mathbb{N} \setminus a} = \omega$. For any $a \in \mathcal{I}^+$, let 1_a denote the characteristic function. Therefore, for any $\rho \in C^*(\mathbb{N})$, $\rho \cdot 1_a$ is a function which is constantly 0 on $\mathbb{N} \setminus a$.

Remark 2.1. It is well-known that, in models of CH, a continuous linear lifting H of $C(\overline{\bigcup_n A_n^*})$ into $C(\mathbb{N}^*)$ need not have the property that $H(f) \cdot H(g) = 0$ whenever $f \cdot g = 0$. This is similar to the fact that it is nearly immediate that if H is a linear isomorphism between function spaces $C(X)$ and $C(Z)$, for X, Z compact and X zero-dimensional, and if H satisfies that $H(f) \cdot H(g) = 0$ whenever f and g are characteristic functions of disjoint clopen sets, then X and Z are homeomorphic. On the other hand, Miljutin[7] proved the surprising fact that $C(2^\omega)$ is linearly isomorphic to $C([0,1])$ (for example).

One quite incomplete step in the outline of the proof in the paper [4] is connected to this aspect of linear isomorphisms. Conditions (2.5) and (2.6) on Page 307 of [4] seem to be essentially making this assumption about the isomorphism H discussed. For example, it is very hard to see how to fulfill property (2.6) without having shown that if $\chi_0 \cdot F = 0$, then $H(\chi_0) \cdot H(F) = 0$.

Comments on the proof: Many readers will know of Shelah's original method [10] for making an existing non-trivial automorphism of $\mathcal{P}(\mathbb{N})/fin$ non-extendable in a generic extension. An almost disjoint family $\{a_\alpha : \alpha \in \omega_1\}$ of infinite subsets of $\mathbb{N}$ is constructed together with a family $\{b_\alpha : \alpha \in \omega_1\}$ of partitioners (i.e. $b_\alpha \subset a_\alpha$) in such a way that there is a ccc poset $\mathbb{P}_{\langle a_\alpha, b_\alpha : \alpha \in \omega_1\rangle}$ which forces the existence of a uniformizing partition X satisfying that $X \cap a_\alpha =^* b_\alpha$ for each $\alpha \in \omega_1$ while preserving that there is no similar uniformizing Y for the family $\{\varphi(a_\alpha), \varphi(b_\alpha) : \alpha \in \omega_1\}$ (because it will contain a Hausdorff-Luzin type of gap). Clearly any possible value for $\varphi(X)$ must be such a uniformizing Y. The set-theoretic principle $\diamondsuit$ is used to help ensure that the poset is ccc. Our method in this paper is based on this approach. We intend to similarly choose a sequence of sets $\{a_\alpha : \alpha \in \omega_1\} \subset \mathcal{J}$ and replace choosing b_α (or rather 1_{b_α}) by choosing some $f_\alpha \in C^*(\mathbb{N})$ with support contained in a_α (i.e. $f_\alpha \cdot 1_{\mathbb{N}\setminus a_\alpha} = 0$) and again making these choices in such a way that we can force the existence of a uniformizing function f_{ω_1} in the sense that $f_{\omega_1} \cdot 1_{a_\alpha}$ asymptotically agrees with f_α for all $\alpha \in \omega_1$. However, the main new obstacle is that while $\varphi(b_\alpha)$ has no interaction with $\varphi(a_\beta)$ for $\beta \neq \alpha$, as remarked above, this is very much not the case with $T(f_\alpha) \cdot T(1_{a_\beta})$. This makes it seemingly impossible to control for the possible existence of a function g which might take the value for $T(f_{\omega_1})$. That is, there is no expectation that $T(f_{\omega_1}) \cdot T(1_{a_\beta})$ should have any sort of clear relationship to $T(f_\beta) \cdot T(1_{a_\beta})$. To handle this we first prove (Lemma 2) the existence of "T-orthogonal pairs" a, c, subsets of $\mathbb{N}$, satisfying that $T(\rho \cdot 1_c) \cdot 1_a$ converges to 0 for all $\rho \in C^*(\mathbb{N})$. After proving the existence of such T-orthogonal pairs, we describe the construction of the poset $\mathbb{P}_{\langle f_\alpha, d_\alpha : \alpha \in \omega_1\rangle}$ (where for other technical reasons $\langle d_\alpha : \alpha \in \omega_1\rangle$ is a mod finite increasing sequence and the above mentioned a_α is contained in $d_{\alpha+1} \setminus d_\alpha$). While constructing this family, we also build in the construction of a suitable Hausdorff-Luzin type gap canonically coded by the family $\langle T(f_{\alpha+1}) : \alpha \in \omega_1\rangle$ which will serve as the device for ensuring that no value for $T(f_{\omega_1})$ will exist. The paper finishes with the necessary lemmas to show that the construction can be carried out.

Let C_1 be the set of functions from $\mathbb{N}$ into $\{-1, 0, 1\}$, and let C_1^+ denote the set of functions from $\mathbb{N}$ into $\{0, 1\}$. For any function $\rho \in C_1$, let ρ^+, ρ^- be the unique members of C_1^+ such that $\rho = \rho^+ - \rho^-$ and $|\rho| = \rho^+ + \rho^-$.

Lemma 2. *Given $a, c \in \mathcal{I}^+$, there are $a_1, c_1 \in \mathcal{I}^+$ such that $a_1 \subset a$, $J_{a_1} = J_a$, $c_1 \subset c$, and for all $\rho \in C^*(\mathbb{N})$, $(T(\rho \cdot 1_{a_1})) \cdot 1_{c_1}$ converges to* 0.

Proof. We may assume that $a \cap c$ is empty. Since we are assuming that T is a lifting, let us note that for all $\rho \in C_1$, there is a $B \in \{A_n : n \in \omega\}^\perp$ such that $T(\rho \cdot 1_a) \cdot 1_{\omega\setminus(a\cup B)}$ converges to 0. In particular then we have that $T(\rho \cdot 1_a) \cdot 1_{c\setminus B}$ converges to 0. This also implies that $T(\rho \cdot 1_a) \cdot 1_c$ is asymptotically equal to $T(\rho \cdot 1_a) \cdot 1_{c\setminus\bigcup_{j<n} A_j}$ for each $n \in \omega$.

Let $\mathcal{L}$ denote the set of pairs (a_1, c_1) satisfying that $a_1 \subset a$, $c_1 \subset c$, $J_{a_1} = J_a$, and $c_1 \in \mathcal{I}^+$. For each $(a_1, c_1) \in \mathcal{L}$, let the real number L_{a_1,c_1} denote the least upper bound of the aymptotic norms of each member of the family $\{T(\rho \cdot 1_{a_1}) \cdot 1_{c_1} : \rho \in C_1\}$. Also let $L^\downarrow_{a_1,c_1} = \inf\{L_{a_2,c_2} : (a_2, c_2) \in \mathcal{L} \text{ and } a_2 \subset a_1, c_2 \subset c_1\}$.

Claim 1. There is a pair $(a_1, c_1) \in \mathcal{L}$ such that $L_{a_1,c_1} = L^\downarrow_{a_1,c_1}$.

Proof of Claim. Let $(a_0, c_0) = (a, c)$ and recursively choose a pairwise descending sequence $\{(a_n, c_n) : n \in \omega\} \subset \mathcal{L}$ so that $L_{a_{n+1},c_{n+1}} < L^\downarrow_{a_n,c_n} + \frac{1}{2^n}$. Notice that for each n, we have that $L^\downarrow_{a_n,c_n} \leq L^\downarrow_{a_{n+1},c_{n+1}} \leq L_{a_{n+1},c_{n+1}} \leq L_{a_n,c_n}$. Choose any set $a_\omega \subset \bigcup_{j\in J_a} A_j$ so that $J_{a_\omega} = J_a$ and for each $j \in J_a$, $a_\omega \cap A_j \subset a_j$ and for each n,

$a_\omega \cap A_j \subset^* a_n$. Notice that $a_\omega \setminus a_n$ is finite for all n. Choose a strictly increasing sequence $\{i_n : n \in \omega\}$ so that for each n, $c_n \cap A_{i_n}$ is infinite. Set $c_\omega = \bigcup_{n\in\omega} c_n \cap A_{i_n}$. We have that $(a_\omega, c_\omega) \in \mathcal{L}$, and that $c_\omega \setminus c_n \subset \bigcup_{i<i_n} A_i$ for all n.

Let ρ be any member of C_1 and let $n \in \omega$. We have that $\rho \cdot 1_{a_\omega}$ is mod finite equal to $(\rho \cdot 1_{a_\omega}) \cdot 1_{a_n}$. Therefore $T(\rho \cdot 1_{a_\omega})$ is asymptotically equal to $T(\ (\rho \cdot 1_{a_\omega}) \cdot 1_{a_n}\ )$. Since the asymptotic norm of $T(\rho \cdot 1_{a_n}) \cdot 1_{c_\omega}$ is less than or equal to that of $T(\rho \cdot 1_{a_n}) \cdot 1_{c_n}$, we have that the asymptotic norm of $T(\rho \cdot 1_{a_\omega}) \cdot 1_{c_\omega}$ is bounded above by each L_{a_n,c_n}. By similar reasoning, it follows that $L^{\downarrow}_{a_\omega,c_\omega}$ is bounded below by $L^{\downarrow}_{a_n,c_n}$ for each n. This completes the proof of the Claim. □

Now that we have proven Claim 1, we may simply assume that $L = L_{a,c}$ is equal to L_{a_1,c_1} for all $(a_1, c_1) \in \mathcal{L}$.

Claim 2. Suppose that (a_1, c_1) and (a_2, c_2) are in $\mathcal{L}$ and that $a_1 \cap a_2$ is finite. Suppose also that ρ_1, ρ_2 are in C_1 and that for some $b \subset c_1$ and some $\epsilon > 0$, the sequence $\{|T(\rho_1 \cdot 1_{a_1})(k)| : k \in b\}$ has no values below $L - \epsilon$. Then the asymptotic norm of the function $T(\rho_2 \cdot 1_{a_2}) \cdot 1_b$ is at most ϵ.

Proof of Claim. Since $(a_1 \cup a_2, c_1)$ is in $\mathcal{L}$ and a_1 and a_2 are disjoint, we have that each of $T(\rho_1 \cdot 1_{a_1} + \rho_2 \cdot 1_{a_2}) \cdot 1_{c_1}$ and $T(\rho_1 \cdot 1_{a_1} - \rho_2 \cdot 1_{a_2}) \cdot 1_{c_1}$ have norm at most L. We also have that each of $(T(\rho_1 \cdot 1_{a_1}) + T(\rho_2 \cdot 1_{a_2})) \cdot 1_b$ and $(T(\rho_1 \cdot 1_{a_1}) - T(\rho_2 \cdot 1_{a_2})) \cdot 1_b$ have norm at most L. The conclusion is then obvious. □

The sets C_1 and C_1^+ will be given the usual finite agreement topologies.

Claim 3. For each $(a_1, c_1) \in \mathcal{L}$ and each $\epsilon > 0$, the set of $\rho \in C_1$ such that $T(\rho \cdot 1_{a_1}) \cdot 1_{c_1}$ has norm greater than $L - \epsilon$ is non-meager.

Proof of Claim. Choose any $\epsilon > 0$ and assume that $\{U_n : n \in \omega\}$ is a descending family of dense open subsets of C_1. There is a strictly increasing sequence $\{k_n : n \in \omega\} \subset \omega$ and functions $t_n : [k_n, k_{n+1}) \to \{0, 1\}$ with the property that, for all $s \in \{0, 1\}^{k_n}$, the basic clopen set $[s \cup t_n]$ is contained in U_n. We additionally require that $[k_n, k_{n+1}) \cap A_j$ is not empty for each $j \in J_a \cap n$. Let $a_2 = \bigcup_n [k_{2n}, k_{2n+1})$ and note that $a_3 = a \setminus a_2$ satisfies that $J_{a_3} = J_a$.

Let $\rho_2 \in C_1$ be any function such that $t_{2n} \subset \rho_2$ for all n. Observe that for all $\psi \in C_1$, the function $\rho_2 \cdot 1_{a_2} + \psi \cdot 1_{a_3}$ is in U_n for each n. Choose $B \in \{A_n : n \in \omega\}^\perp$ so that $T(\rho_2 \cdot 1_{a_2}) \cdot 1_{c_1 \setminus B}$ converges to 0. Choose $\psi \in C_1$ so that $T(\psi \cdot 1_{a_3}) \cdot 1_{c \setminus B}$ has norm greater than $L - \epsilon$. Finish the proof of the claim by observing that $T(\rho_2 \cdot 1_{a_2} + \psi \cdot 1_{a_3}) \cdot 1_{c_1 \setminus B}$ is asymptotically equal to $T(\psi \cdot 1_{a_3}) \cdot 1_{c \setminus B}$ and so has norm greater than $L - \epsilon$. □

Next we want to separate the contributions of ρ^+ and ρ^- to the norm of $T(\rho \cdot 1_{a_1}) \cdot 1_{c_1}$. Consider any $\rho \in C_1$ and $(a_1, c_1) \in \mathcal{L}$ and let L_ρ denote the norm of $T(\rho \cdot 1_{a_1}) \cdot 1_{c_1}$. Let $\mathcal{B}^+(\rho, a_1, c_1)$ denote the collection of infinite sets (if any) $b \subset c_1$ such that $T(\rho \cdot 1_{a_1}) \restriction b$ converges to L_ρ. Similarly let $\mathcal{B}^-(\rho, a_1, c_1)$ denote the collection of infinite sets $b \subset c_1$ such that $T(\rho \cdot 1_{a_1}) \restriction b$ converges to $-L_\rho$. We will identify four types of possible behavior. When $\mathcal{B}^+(\rho, a_1, c_1)$ is non-empty we will identify type 1 and type 2. The case when $\mathcal{B}^+(\rho, a_1, c_1)$ is empty will be categorized as type 3 or type 4. It will be completely symmetric in that if ρ is type 3 or type 4, then $-\rho$ will be type 1 or type 2 respectively.

Let us focus on the case when $\mathcal{B}^+(\rho, a_1, c_1)$ is non-empty. We define $v(\rho, a_1, c_1)$ connected to $T(\rho^+ \cdot 1_{a_1})$. Define $v(\rho, a_1, c_1)$ to be the supremum of the norms of the family $\{T(\rho^+ \cdot 1_{a_1}) \cdot 1_b : b \in \mathcal{B}^+(\rho, a_1, c_1)\}$. Similarly define $w(\rho, a_1, c_1)$ to be the supremum of the norms of the family $\{T(\rho^- \cdot 1_{a_1}) \cdot 1_b : b \in \mathcal{B}^+(\rho, a_1, c_1)\}$. Notice that $L_\rho \le v(\rho, a_1, c_1) + w(\rho, a_1, c_1)$, and so $\max(v(\rho, a_1, c_1), w(\rho, a_1, c_1)) \ge \frac{L_\rho}{2}$. We will categorize ρ as type 1 for (a_1, c_1), when $v(\rho, a_1, c_1) \ge \frac{L_\rho}{2}$.

Clearly, for each $(a_1, c_1) \in \mathcal{L}$ and each $\epsilon > 0$, there is a non-meager set of ρ with $L_\rho > L - \epsilon$ of one of the four types for (a_1, c_1). Let $\mathcal{L}_i$ denote the set of $(a_1, c_1) \in \mathcal{L}$ for which, for each $\epsilon > 0$, there is a non-meager set of ρ with $L_\rho > L - \epsilon$ which is type i for (a_1, c_1). By redefining (a, c) to be some member of $\mathcal{L}_i$, we may assume that for each $(a_1, c_1) \in \mathcal{L}$, there is an $(a_2, c_2) \in \mathcal{L}_i$ with $a_2 \subset a_1$ and $c_2 \subset c_1$. For the remainder of the proof we assume, by symmetry, that this is true of $\mathcal{L}_1$.

This leads to the next claim, and the conclusion that $\mathcal{L}_1 = \mathcal{L}$.

Claim 4. For each $(a_1, c_1) \in \mathcal{L}_1$ and each $\epsilon > 0$, there is a non-meager set of $\rho \in C_1$ such that there are infinite disjoint b, d contained in c_1 so that

(1) the set $T(\rho \cdot 1_{a_1})[b]$ only has values greater than $L - \epsilon$,
(2) the $T(\rho^+ \cdot 1_{a_1})[b]$ only has values greater than $\frac{L}{2} - \epsilon$,
(3) the set $T(-\rho \cdot 1_{a_1})[d]$ only has values greater than $L - \epsilon$.
(4) the set $T(\rho^- \cdot 1_{a_1})[d]$ only has values greater than $\frac{L}{2} - \epsilon$.

Proof of Claim. Choose any $\epsilon > 0$ and assume that $\{U_n : n \in \omega\}$ is a descending family of dense open subsets of C_1. Choose a strictly increasing sequence $\{k_n : n \in \omega\} \subset \omega$ and functions $t_n : [k_n, k_{n+1}) \to \{-1, 0, 1\}$ so that, for all $s \in \{-1, 0, 1\}^{k_n}$, the basic clopen set $[s \cup t_n]$ is contained in U_n. We again require that $[k_n, k_{n+1}) \cap A_j \cap a_1$ is not empty for each $j \in J_a \cap n$. Let $a_2 = \bigcup_n [k_{2n}, k_{2n+1})$ and choose disjoint $a_3, a_4 \subset a_1 \setminus a_2$ so that $J_{a_3} = J_{a_4} = J_a$.

Let $\rho_2 \in C_1$ be any function such that $t_{2n} \subset \rho_2$ for all n. Observe that for all $\psi \in C_1$, the function $\rho_2 \cdot 1_{a_2} + \psi \cdot 1_{a_3 \cup a_4}$ is in U_n for each n. Choose $B_0 \in \{A_n : n \in \omega\}^\perp$ so that each of $T(\rho_2 \cdot 1_{a_2} \cdot 1_{a_1})$, $T(\rho_2^+ \cdot 1_{a_2} \cdot 1_{a_1})$, and $T(\rho_2^- \cdot 1_{a_2} \cdot 1_{a_1})$ converges to 0 on the set $c_1 \setminus B_0$. By shrinking a_3 we may suppose that there is some $c_3 \subset c_1 \setminus B_0$ so that $(a_3, c_3) \in \mathcal{L}_1$. Therefore we can choose $\psi_3 \in C_1$ and some $b \in \mathcal{B}^+(\psi_3, a_3, c_3)$ so that the function $T(\psi_3^+ \cdot 1_{a_3})$ only has values greater than $\frac{L}{2} - \frac{\epsilon}{4}$ on the set b.

Now choose $B_1 \in \{A_n : n \in \omega\}^\perp$ containing B_0 so that each of $T(\rho_2 \cdot 1_{a_2} \cdot 1_{a_1} + \psi_3 \cdot 1_{a_3}) \cdot 1_{c_1 \setminus B_1}$, $T((\rho_2 \cdot 1_{a_2} \cdot 1_{a_1} + \psi_3 \cdot 1_{a_3})^+) \cdot 1_{c_1 \setminus B_1}$, and $T((\rho_2 \cdot 1_{a_2} \cdot 1_{a_1} + \psi_3 \cdot 1_{a_3})^-) \cdot 1_{c_1 \setminus B_1}$ converges to 0.

Similarly, by shrinking a_4, choose a function $\psi_4 \in C_1$ and an infinite set $d \subset c_1 \setminus B_1$ so that the image of d by $T(\psi_4 \cdot 1_{a_4})$ has no values below $L - \epsilon$, and the image of d by $T(\psi_4^+ \cdot 1_{a_4})$ has no values below $\frac{L}{2} - \epsilon$.

Now set $\rho = \rho_2 \cdot 1_{a_2} + \psi_3 \cdot 1_{a_3} - \psi_4 \cdot 1_{a_4}$ which is a member of the dense G_δ set $\bigcap_n U_n$. By the choice of B_1 and the linearity of T, we have that $T(\rho \cdot 1_{a_1}) = T(\rho_2 \cdot 1_{a_2} \cdot 1_{a_1} + \psi_3 \cdot 1_{a_3} - \psi_4 \cdot 1_{a_4})$ asymptotically agrees with $T(-\psi_4 \cdot 1_{a_4})$ on d. Similarly $T(\rho^- \cdot 1_{a_1})$ asymptotically agrees with $T(\psi_4 \cdot 1_{a_4})$ on d. This proves that items (3) and (4) of the Claim hold.

By Claim 2, we have that each of $T(\psi_4 \cdot 1_{a_4})$ and $T(\psi_4^- \cdot 1_{a_4})$ converge to 0 along b. We also have that $T(\rho_2 \cdot 1_{a_2} \cdot 1_{a_1} + \psi_3 \cdot 1_{a_3})$ asymptotically agrees with $T(\psi_3 \cdot 1_{a_3})$ along b; and $T(\rho_2^+ \cdot 1_{a_2} \cdot 1_{a_1} + \psi_3^+ \cdot 1_{a_3})$ asymptotically agrees with $T(\psi_3^+ \cdot 1_{a_3})$ along b. Putting all this together we have that $T(\rho \cdot 1_{a_1})$ asymptotically agrees with $T(\psi_3 \cdot 1_{a_3})$ along b, and $T(\rho^+ \cdot 1_{a_1})$ asymptotically agrees with $T(\psi_3^+ \cdot 1_{a_3})$ along b. This verifies items (1) and (2) of the Claim. □

Now we are ready to apply OCA arguments to continue the proof. For each $j \in J_a$, choose any injection ψ_j from $2^{<\omega}$ into $a \cap A_j$. Also choose, for each $j \in J_c$, an injection σ_j of J_c into $A_j \cap c$. For each $r \in 2^\omega$, let a_r denote the set $a_r = \{\psi_j(r \restriction \ell) : j < \ell \in \omega\}$.

Let $\mathcal{X}$ denote the collection of functions of the form $\rho = \rho \cdot 1_{a_r}$ for some $r \in 2^\omega$, and $\rho \in C_1$ so that Claim 4 holds for some pair $b, d \subset c$.

For $\rho \in \mathcal{X}$, let

$$b_\rho = \{k \in c : T(\rho^+)(k) > .45L \text{ and } T(\rho)(k) > .9L\}$$

and

$$d_\rho = \{k \in c : T(\rho^-)(k) > .45L \text{ and } T(\rho)(k) < -.9L\}$$

We define an open relation K_0 on $[\mathcal{X}]^2$ as follows. A pair $(\rho_r, \rho_s) \in K_0$ providing

(1) $r \neq s$ are members of 2^ω,
(2) $\rho_r \cdot 1_{a_r} = \rho_r$,
(3) $\rho_s \cdot 1_{a_s} = \rho_s$,
(4) ρ_r and ρ_s agree on $a_r \cap a_s$,
(5) there is a $k \in (b_{\rho_r} \cap d_{\rho_s}) \cup (b_{\rho_s} \cap d_{\rho_r})$

Assume that $\{\rho_\alpha : \alpha \in \omega_1\} \subset C_1$ and $\{r_\alpha : \alpha \in \omega_1\} \subset 2^\omega$ are such that $[\{\rho_\alpha \cdot 1_{a_{r_\alpha}} : \alpha \in \omega_1\}]^2$ is contained in K_0. For each α, let $a_\alpha = a_{r_\alpha}$ and assume, with no loss, that $\rho_\alpha = \rho_\alpha \cdot 1_{a_\alpha}$. For each α, let $b_\alpha = b_{\rho_\alpha}$ and $d_\alpha = d_{\rho_\alpha}$ Because of (5) the family $\{(b_\alpha, d_\alpha) : \alpha \in \omega_1\}$ forms a Luzin family and so there is no set $Y \subset \mathbb{N}$ and uncountable $\Gamma \subset \omega_1$ such that Y mod finite separates the family $\{b_\alpha : \alpha \in \Gamma\}$ from the family $\{d_\alpha : \alpha \in \Gamma\}$.

We consider the functions f^+, f^- where, for each k,

$$f^+(k) = \max\{\rho_\alpha^+(k) : \alpha \in \omega_1\} \text{ and } f^-(k) = \max\{\rho_\alpha^-(k) : \alpha \in \omega_1\} .$$

Also let $f = f^+ - f^-$. Notice that $f \cdot 1_a = f$ and, for each $\alpha \in \omega_1$, $f \cdot 1_{a_\alpha} = \rho_\alpha$.

Claim 5. The liminf of $T(f)$ on b_α is at least $.8L$

Proof of Claim. Assume that b is any infinite subset of b_α and assume that $T(f) \restriction b$ converges to some L_b. By thinning b we may also assume that each of $T(\rho_\alpha) \restriction b$ and $T(f \cdot 1_{a \setminus a_\alpha}) \restriction b$ also converge. We know that $T(\rho_\alpha) \restriction b$, converges to some value greater than or equal to $.9L$. By Claim 2, $T(f \cdot 1_{a \setminus a_\alpha}) \restriction b$ must converge to values with absolute value less than or equal to $.1L$. □

Similarly, we have

Claim 6. The limsup of $T(f)$ on d_α is at most $-.8L$.

Now that we have that $Y = T(f)^{-1}(0, \infty)$ will mod finite separate the entire family $\{b_\alpha : \alpha \in \omega_1\}$ from $\{d_\alpha : \alpha \in \omega_1\}$, there is evidently no such uncountable K_0-homogeneous set.

Therefore, by OCA, we deduce there is a countable family $\{\mathcal{Y}_n : n \in \omega\}$ which covers $\mathcal{X}$ with the property that $[\mathcal{Y}_n]^2 \cap K_0$ is empty for all n. For each n, there is a countable Y_n which is a dense subset of $\mathcal{Y}_n$ in the suitable metric topology inherited from $\mathcal{X}$.

Choose any selective ultrafilter $\mathcal{U}$ on ω such that $J_c \in \mathcal{U}$. For each $U \in \mathcal{U}$, let $\sigma[U]$ denote the set $\{\sigma_j(k) : j, k \in U \cap J_c \text{ and } |U \cap k| > j\}$. The family $\{\sigma[U] : U \in \mathcal{U}\}$ is a base for an ultrafilter on $\mathbb{N}$. It is the $\mathcal{U}$-limit of the sequence $\{\sigma_j(\mathcal{U}) : j \in J_c\}$. To see this, assume that $W \subset \mathbb{N}$ is such that $U_W = \{j \in J_c : \sigma_j^{-1}(W \cap A_j) = U_j \in \mathcal{U}\} \in \mathcal{U}$.

Since $\mathcal{U}$ is selective, there is a $U \in \mathcal{U}$ such that $U \subset U_W$ and, for each $j \in U$, $U \setminus \bigcap_{\ell<j} U_\ell$ has cardinality less than j. It follows that $\sigma[U] \subset W$.

Fix any countable elementary submodel with each of the above objects as elements. For $\eta \in \mathcal{X}$, let r_η denote the member of 2^ω such that $\eta \cdot 1_{a_{r_\eta}} = \eta$. We will choose an $r \in 2^\omega$ and then recursively define a $\rho \in C_1$ with $\rho \cdot 1_{a_r} = \rho$.

Let us consider any $s \in 2^{<\omega}$ and let $a^s = \{\psi_j(s \restriction m) : j < m < |s|\}$ which is the maximal common initial segment of a_r for all $s \subset r \in 2^\omega$. Also fix any $\rho_s : a^s \to \{-1, 0, 1\}$. For any $n \in \omega$ and $s \subset t \in 2^{<\omega}$, define $\rho_{s,t} \supset \rho_s$ to be the function with domain a^t which has value 0 on $a^t \setminus a^s$. We will consider the two sets

$$W(\rho_s, n, t, 0) = \{k \in c : (\exists \eta \in \mathcal{Y}_n)\rho_{s,t} \subset \eta, t \subset r_\eta \text{ and } T(\eta)(k) > .9L\}$$

and

$$W(\rho_s, n, t, 1) = \{k \in c : (\exists \eta \in \mathcal{Y}_n)\rho_{s,t} \subset \eta, t \subset r_\eta \text{ and } T(\eta)(k) < -.9L\}\ .$$

There is a sequence $\{U_m : m \in \omega\} \subset \mathcal{U}$ such that for each such s, ρ_s, n, t, there is an m such that $\sigma[U_m]$ is either contained in, or disjoint from, $W(\rho_s, n, t, 0) \cap W(\rho_s, n, t, 1)$. Fix any $U \in \mathcal{U}$ which is mod finite contained in each U_m.

Choose any $r \in 2^\omega$ with the property that it does not contain any infinite chain of the form $E_{n,s,\rho_s} = \{t \in 2^{<\omega} : s \subset t, \text{ and } W(\rho_s, n, t, 0) \cap W(\rho_s, n, t, 1) \in \mathcal{U}\}$ where $s \in \{r \restriction \ell : \ell \in \omega\}$, $n \in \omega$, and $\rho_s : a^s \to \{-1, 0, 1\}$. In other words, if such an E_{n,s,ρ_s} is contained in r, then it is finite. Since there are only countably many such chains, there is such an r.

Consider the forcing $\mathbb{P}_r$ consisting of finite approximations $\rho_s : a^s \to \{-1, 0, 1\}$ to a generic function $\rho : a_r \to \{-1, 0, 1\}$. Since $(a_r, \sigma[U]) \in \mathcal{L}_1$, whenever $\mathcal{D}$ is a countable family of dense subsets of $\mathbb{P}_r$, there will be a non-meager set of $\mathcal{D}$-generic ρ that will satisfy that, not only is $\rho \in \mathcal{X}$, but also that b_ρ and d_ρ each hit $\sigma[U]$ in an infinite set.

Now for each integer n, define

$$D_n = \{\rho_{s,t} \in \mathbb{P}_r : \text{ either } t \notin E_{n,s,\rho_s} \text{ or } (\exists \bar{t} \in E_{n,s,\rho_s})(s \subset \bar{t} \perp t)\}\ .$$

Fix any $\rho_s \in \mathbb{P}_r$. If E_{n,s,ρ_s} is a chain, there is an extension $s \subset t \subset r$ such that $t \notin E_{n,s,\rho_s}$. Therefore $\rho_{s,t} \in D_n$. Otherwise, there is an extension $\bar{t} \supset s$ such that $\bar{t} \not\subset r$ and $\bar{t} \in E_{n,s,\rho_s}$. Choose any $s \subset t \subset r$ such that $t \perp \bar{t}$. Then we have that $\rho_{s,t} \in D_n$. This shows that D_n is dense.

Now we assume that ρ is $\{D_n : n \in \omega\}$-generic over $\mathbb{P}_r$ and that $\rho \in \mathcal{X}$ and that each of b_ρ and d_ρ meet $\sigma[U]$ in an infinite set. Notice also that b_ρ and d_ρ necessarily meet each $c \cap A_j$ in a finite set. Therefore, $b_\rho \cap \sigma[U]$ and $d_\rho \cap \sigma[U]$ are mod finite contained in $\sigma[U_m]$ for each m. Consider any n and assume that $\rho \in \mathcal{Y}_n$. By the density of D_n, there is an $s \subset t \subset r$ such that $\rho_{s,t} \subset \rho$ and $\rho_{s,t} \in D_n$. Choose the $\bar{t} \perp t$ so that $\rho_{s,\bar{t}} \in E_{n,s,\rho_s}$. Since there is an m such that $\sigma[U_m] \subset W(\rho_s, n, \bar{t}, 0)$, there is a $k \in d_\rho \cap W(\rho_s, n, \bar{t}, 0)$. Choose $\eta \in \mathcal{Y}_n$ so that $\rho_{s,\bar{t}} \subset \eta$, $t \subset r_\eta$, and $T(\eta)(k) > .9L$. We have now produced $\rho, \eta \in \mathcal{Y}_n$ such that $\{\rho, \eta\} \in K_0$.

This completes the proof of the Lemma. □

Let us say that a set a is T-orthogonal to a set c if for all $\rho \in C_1$, $T(\rho \cdot 1_c) \cdot 1_a$ converges to 0. So far as we know, this is not a symmetric relation. Although it does follow from Lemma 2 that there are mutually T-orthogonal pairs, we do not know if there is such a choice with $c \in \mathcal{J}$ (as we will need), and so we are satisfied with the asymmetry.

Following a standard method of producing a proper poset for the application of PFA we pass to the CH extension obtained by forcing with $\omega_2^{<\omega_1}$. For any $h \in C_1$ and $d \in \mathcal{I}^+$, we define the poset $P_{h,d}$ to be the set of partial functions p from $\mathbb{N}$ into $\{-1,0,1\}$ such that $\mathrm{dom}(p) \subset^* d$ and $p \subset^* h$ (in the sense of only finitely many disagreements). For any $\alpha \leq \omega_1$ and sequence $\langle f_\beta, d_\beta : \beta < \alpha\rangle$ of such $f_\beta \in C_1$ and $d_\beta \in \mathcal{I}^+$, satisfying that for $\beta < \gamma < \alpha$ $d_\beta \subset^* d_\gamma$ and $f_\gamma \cdot 1_{d_\beta} =^* f_\beta \cdot 1_{d_\beta}$, the poset $P_{\langle f_\beta, d_\beta : \beta<\alpha\rangle}$ is defined to be $\bigcup_{\beta<\alpha} P_{f_\beta,d_\beta}$ We can fix a $\diamondsuit$-sequence $\{S_\alpha : \alpha \in \omega_1\} \subset [\omega_1]^{\leq\omega}$ and fix an enumeration $\{H_\alpha : \alpha \in \omega_1\}$ of $H(\omega_1)$ (the hereditarily countable sets).

Now we define a sequence $\{d_\beta, f_\beta, \rho_\beta, a_\beta, \mathcal{D}_\beta : \beta \in \omega_1\}$ subject to the following inductive assumptions on α: for $\beta < \gamma < \alpha$,

(1) $d_\gamma \in \mathcal{J}$,
(2) $a_\beta \subset d_\gamma$, $a_\beta \in \mathcal{I}^+$, and $d_\beta \cup a_\beta \subset d_{\beta+1}$,
(3) $f_\beta, f_\gamma \in C_1$ and $f_\beta = f_\beta \cdot 1_{d_\beta} =^* f_\gamma \cdot 1_{d_\beta}$,
(4) $f_\gamma \cdot 1_{a_\beta} = \rho_\beta$
(5) for all $\rho \in C_1$, $T(\rho \cdot 1_{\mathbb{N}\setminus d_\gamma}) \cdot 1_{a_\beta}$ converges to 0
(6) $\mathcal{D}_\beta$ is a countable family of predense subsets of P_{f_β,d_β}
(7) $\mathcal{D}_\beta \subset \mathcal{D}_\gamma$,
(8) if $D_\gamma = \{H_\xi : \xi \in S_\gamma\}$ is a predense subset of $P_{\langle f_\beta,d_\beta:\beta<\gamma\rangle}$, then $D_\gamma \in \mathcal{D}_\gamma$

This construction using $\diamondsuit$ as in condition (8) will ensure that the poset $P_{\omega_1} = P_{\langle f_\beta,d_\beta:\beta<\omega_1\rangle}$ is ccc. This is from Shelah's oracle chain condition method of [10, §IV]. We also work with a listing, $\{\dot{Y}_\beta : \beta < \omega_1\}$, of all nice P_{ω_1}-names of subsets of $\mathbb{N}$ such that $\dot{Y}_\beta$ is a P_{f_γ,d_γ}-name (for any $\beta < \gamma$). And we add the inductive condition

(9) for $\beta < \gamma$, $P_{\langle f_\xi,d_\xi:\xi<\gamma\rangle}$ forces that $\dot{Y}_\beta$ does not mod finite separate b_γ from e_γ where $b_\gamma = \{k \in a_\gamma : T(f_{\gamma+1})(k) > \frac{2}{3}\}$ and
$e_\gamma = \{k \in a_\gamma : T(f_{\gamma+1})(k) < \frac{1}{3}\}$.

After constructing a_γ and ρ_γ, we are able to preserve the property in item (9) by adding a specific countable family of dense sets to $\mathcal{D}_{\gamma+1}$.

The construction of this sequence will be explained in a series of Lemmas. However before doing so, we indicate how this will prove the main theorem. After forcing with P_{ω_1}, we have that the family $\{b_\gamma, e_\gamma : \gamma \in \omega_1\}$ can not be σ-separated. This implies ([13, Theorem 2] and [11, Lemma 2]) there is a proper poset Q which introduces an uncountable $\Gamma \subset \omega_1$ so that the family $\{b_\gamma, e_\gamma : \gamma \in \Gamma\}$ is a Luzin family (it is unsplit in any proper forcing extension). Now, we meet ω_1 many dense subsets of $\omega_2^{<\omega_1} * P_{\omega_1} * Q$ in order to decide on the generic function $f = f_{\omega_1}$ added by P_{ω_1}, and the Luzin gap $\{b_\gamma, e_\gamma : \gamma \in \Gamma\}$ as well as the basic properties of the family as detailed in items (1) - (6). Notice that (by the inclusion of ω_1 many dense subsets of P_{ω_1}) $f \cdot 1_{d_\gamma}$ is almost equal to f_γ. It follows then that $T(f)$ can not exist. This is because $Y = T(f)^{-1}((\frac{1}{2},\infty))$ is required to split the Luzin gap. To see this we have to show that $T(f)\cdot 1_{b_\gamma}$ has liminf greater than $\frac{1}{2}$, while $T(f) \cdot 1_{e_\gamma}$ has limsup less than $\frac{1}{2}$. We consider $T(f) \cdot 1_{a_\gamma}$ as asymptotically equal to $T(f \cdot 1_{d_{\gamma+1}}) \cdot 1_{a_\gamma} + T(f \cdot 1_{\mathbb{N}\setminus d_{\gamma+1}}) \cdot 1_{a_\gamma}$. Items (3) and (5) ensure that this is asymptotically equal to $T(f_{\gamma+1}) \cdot 1_{a_\gamma}$. Therefore, $Y \cap a_\gamma$ does separate b_γ and c_γ.

We construct, by induction on $\alpha \in \omega_1$, the sequences

$$\langle f_\beta, d_\beta, \mathcal{D}_\beta : \beta < \alpha\rangle \cup \langle \rho_\beta, a_\beta : \beta + 1 < \alpha\rangle$$

as per inductive items (1)-(9) above. We can start very simply with $d_0 = \emptyset$, f_0 the constant 0 function, and $\mathcal{D}_0 = \{\emptyset\}$.

If α is a limit ordinal, then the choices of f_α, d_α and $\mathcal{D}_\alpha$ are handled at the end in Lemma 6. Therefore, we can proceed by assuming that we have constructed the family

$$\langle f_\beta, d_\beta, \mathcal{D}_\beta : \beta \leq \alpha\rangle \cup \langle \rho_\beta, a_\beta : \beta + 1 \leq \alpha\rangle \ .$$

The choices for $f_{\alpha+1}, d_{\alpha+1}, \mathcal{D}_{\alpha+1}$ together with a_α, ρ_α are established in Lemma 5. We will need preparatory lemmas leading up it.

This next lemma is (essentially) statement (*1) of [10, IV §5, p134]. We sketch a proof for the reader's convenience.

Lemma 3. *Assume that $h \in C_1$ and $d \in \mathcal{J}$ are such that $h \cdot 1_d = h$ and assume that $c \in \mathcal{I}^+$ is disjoint from d. If $\mathcal{E}$ is a countable family of predense subsets of $P_{h,d}$, then there is an $a \subset c$ such that $J_a = J_{a\setminus c} = J_c$ so that for all $\rho \in C_1$ each $E \in \mathcal{E}$ is a predense subset of the poset $P_{h+\rho\cdot 1_a, d\cup a}$.*

Moreover, given c and a as above let $d_1 = d \cup (c \setminus a)$. Then there is an h_1 such that $h_1 \cdot 1_{d_1} = h_1$, $h_1 \cdot 1_d = h \cdot 1_d$, and such that for all $\rho \in C_1$ with $\rho \cdot 1_a = \rho$, each $E \in \mathcal{E}$ is a predense subset of $P_{h_1+\rho, d_1\cup a}$.

Proof. Let $\{p_\ell : \ell \in \omega\}$ enumerate all members finite functions from the poset $P_{h,d}$. Let $p_\ell \oplus h$ denote the function $p_\ell \cup h \restriction (d \setminus \mathrm{dom}(p_\ell))$. Let $\{E_\ell : \ell \in \omega\}$ be a descending sequence of dense subsets of $P_{h,d}$ so that the downward closure of each $E \in \mathcal{E}$ contains one of them. Recursively define an increasing sequence $\langle n_k : k \in \omega\rangle$ of integers as follows. Let $n_0 = 0$ and given n_k ensure that n_{k+1} is large enough so that $\mathrm{dom}(p_\ell) \subset n_{k+1}$ for all $\ell < n_k$ and that there is some $\ell_k < n_{k+1}$ so that $\mathrm{dom}(p_{\ell_k})$ is contained in $[n_k, n_{k+1}) \setminus d$, and, for all ℓ such that $\mathrm{dom}(p_\ell) = n_k$, $p_{\ell_k} \cup (p_\ell \oplus h) \in E_k$. In addition, ensure that $c \cap A_j \cap [n_k, n_{k+1})$ is not empty for each $j \in J_c \cap n_k$.

Let $a = c \cap \bigcup\{[n_{2k}, n_{2k+1}) : k \in \omega\}$. Note that $J_a = J_{c\setminus a} = J_c$. Let ρ be any member of C_1 and fix any $E \in \mathcal{E}$. We check that E is predense in $P_{h+\rho\cdot 1_a, d\cup a}$. To do so we consider any $q \in P_{h+\rho, d\cup a}$. By extending q we may assume that $\mathrm{dom}(q)$ contains $d \cup a$. Choose k large enough so that the downward closure of E in $P_{h,d}$ contains E_k, $\mathrm{dom}(q) \subset d \cup a \cup n_{2k+1}$, and such that $q(j) = (h + \rho)(j)$ for all $n_{2k+1} < j \in d \cup a$. There is an ℓ such that $q \restriction n_{2k+1}$ is contained in p_ℓ and $\mathrm{dom}(p_\ell) = n_{2k+1}$. By construction $p_{\ell_{2k+1}} \cup (p_\ell \oplus h)$ is in E_{2k}. Since $p_{\ell_{2k+1}} \cup (p_\ell \oplus h)$ is contained in $p_{\ell_{2k+1}} \cup q$, we have that q is compatible with a member of E.

Now assume that $d \cup c \in \mathcal{J}$ and choose $h_1 \in C_1$ so that $h_1 \cdot 1_d = h \cdot 1_d$ and so that $h \restriction c = \bigcup\{p_{\ell_k} \restriction c : k \in \omega\}$. Also ensure that $h_1 \cdot 1_{\mathbb{N}\setminus d_1}$ is 0. The same argument as above shows that each $E \in \mathcal{E}$ is predense in P_{h_1,d_1} because $p_k \restriction c \subset h_1$ for all k. □

Having chosen f_α, d_α, we are ready to choose a_α. First apply Lemma 2 to find $\tilde{a}_\alpha \in \mathcal{I}^+$ and disjoint $c_\alpha \subset \mathbb{N} \setminus d_\alpha$ so that $\tilde{a}_\alpha$ is T-orthogonal to c_α and so that $J_{c_\alpha} = \omega$. Next apply Lemma 3 (with $c = \tilde{a}_\alpha$) to choose any $a_\alpha \in \mathcal{I}^+$ contained in $\tilde{a}_\alpha$ and $h_{\alpha,0}$ so that $h_{\alpha,0} \cdot 1_{d_\alpha} = f_\alpha$, $h_{\alpha,0} \cdot 1_{a_\alpha\cup c_\alpha} = 0$ such that we are free to choose any $\rho_\alpha \in C_1$ with $\rho_\alpha = \rho_\alpha \cdot 1_{\tilde{a}_\alpha}$ so as to preserve that each member of the family $\mathcal{D}_\alpha$ is predense in the poset $P_{h_{\alpha,0}+\rho_\alpha, \mathbb{N}\setminus(a_\alpha\cup c_\alpha)}$. Set $d_{\alpha,0} = \mathbb{N} \setminus (a_\alpha \cup c_\alpha)$.

With this reduction, we have now guaranteed that with this choice of a_α and $d_{\alpha+1} = \mathbb{N} \setminus c_\alpha$, then for all $\gamma > \alpha$, so long as $f_\gamma \cdot 1_{d_{\alpha+1}} =^* f_{\alpha+1} = f_{\alpha+1} \cdot 1_{d_{\alpha+1}}$ (as in inductive condition (3)) is satisfied, then $T(f_\gamma) \cdot 1_{a_\alpha}$ will be asymptotically equal to $T(f_{\alpha+1})\cdot 1_{a_\alpha}$. The reason is that $T(f_\gamma) - T(f_{\alpha+1})$ will be asymptotically equal to $T(f_\gamma \cdot 1_{c_\alpha})$, and a_α is T-orthogonal to c_α.

The key property of the choice of ρ_α is the requirement on $\dot{Y}_\beta$ for each $\beta < \alpha$. This next Lemma shows how to handle one such β, then we extend to all countably many in the subsequent Lemma.

Lemma 4. *Let a, d be disjoint members of $\mathcal{I}^+$ and let $h \in C_1$ be such that $h \cdot 1_d = h$. Further suppose that $\dot{Y}$ is a $P_{h,d}$-name for a subset of $\mathbb{N}$ and let p_0 be any member of $P_{h,d}$. Then there is a $\rho \in C_1$ such that, $p_0 \subset \rho$, $\rho \cdot 1_{d\cup a} =^* \rho$, and such that $\rho \restriction (d \cup a)$ forces, with respect to the poset $P_{h+\rho\cdot 1_a, d\cup a}$, that $\dot{Y}$ does not mod finite separate $a \cap T(\rho)^{-1}(\frac{2}{3}, \infty)$ and $a \cap T(\rho)^{-1}(-\infty, \frac{1}{3})$.*

Proof. Assume that $\dot{Y}$ is such a name and that there is no such ρ. Fix any integer L, we will prove that T has norm exceeding L. We may obviously assume that a is disjoint from $\mathrm{dom}(p_0)$ and that $\mathrm{dom}(p_0) \supset d$. We may assume that $\dot{Y}$ is a simple name that is a subset of $\mathbb{N} \times P_{h,d}$ and, for a generic filter G, $\mathrm{val}_G(\dot{Y}) = \{k : (\exists r \in G)(k, r) \in \dot{Y}\}$. Let $p_0{}^\frown\bar{0} \in C_1$ denote the extension of p_0 satisfying that $p_0{}^\frown\bar{0} \cdot 1_{\mathrm{dom}(p_0)} = p_0{}^\frown\bar{0}$. By the properties of T we have that $T(p_0{}^\frown\bar{0})$ converges to 0 on $a \cap A_j$ for each $j \in J_a$. By removing a finite set from each $a \cap A_j$, we may assume that $T(p_0{}^\frown\bar{0})(k)$ has absolute value less than $\frac{1}{9}$ for all $k \in a$.

Fix, for each $j \in J_a$ an injection $\psi_j : 2^{<\omega} \to a \cap A_j$. Our plan is to choose $\rho \in C_1$ so that for all j, $x_{\rho,j} = \{s \in 2^{<\omega} : \rho(\psi_j(s)) \neq 0\}$ is a chain. Let $Q \subset P_{h,d}$ denote the set of those $p \in P_{h,d}$ with this same property, namely, that for all j, $x_{p,j} = \{s \in 2^{<\omega} : p(\psi_j(s)) \neq 0\}$ is a (possibly empty) chain. Let $x^+_{p,j} = \{s \in x_{p,j} : p(\psi_j(s)) > \frac{7}{9}\}$ and $x^-_{p,j} = \{s \in x_{p,j} : p(\psi_j(s)) < \frac{2}{9}\}$. The ordering on Q, inherited from $P_{h,d}$, is that $r \leq_Q q$ providing $q \subseteq r$. We may consider $\dot{Y}$ (equivalently $\dot{Y} \cap (\mathbb{N} \times Q)$) as a Q-name. Fix an enumeration $\{q_\ell : \ell \in \omega\}$ of $\{q \in Q : \mathrm{dom}(q) \cap a = \emptyset\}$.

For any $j \in J_a$, say that an element $q \in Q$ is j-decisive if for all $q \subset r$ in Q, $r \Vdash_Q \psi_j(t) \in \dot{Y}$ for all $t \in x^+_{r,j} \setminus x_{q,j}$, and $r \Vdash_Q \psi_j(t) \notin \dot{Y}$ for all $t \in x^-_{r,j} \setminus x_{q,j}$.

Claim 7. For each $p_0 \subseteq p \in Q$ and $j \in J_a$ there is a $p \subseteq q$ in Q which is j-decisive.

If no such q exists, then, we recursively choose an $\subset$-increasing sequence $\{r_k : k \in \omega\} \subset Q$ with $p = r_0$ and $\mathrm{dom}(r_k \setminus p) \subset a$ for all k. Also ensure that $\bigcup_k \mathrm{dom}(r_k) = a$. The inductive hypothesis is that for each k and each $\ell < k$, if $q_\ell \cup r_k \in Q$, then either there is ℓ' and a $t \in x^+_{r_{k+1},j} \setminus x_{r_k,j}$ such that $q_{\ell'} \cup r_{k+1} \in Q$, $q_{\ell'} \cup r_{k+1} < q_\ell \cup r_k$, and $q_{\ell'} \cup r_{k+1} \Vdash \psi_j(t) \notin \dot{Y}$, or a similar conclusion for some $t \in x^-_{r_{k+1},j} \setminus x_{r_k,j}$.

Upon completion of this recursion, set $\rho = \bigcup_k r_k$. We check that ρ is as required in the conclusion of the Lemma. First of all, let us recall that ρ and $T(\rho)$ are asymptotically equivalent on $a \cap A_j$. So there is an k_0 such that $|\rho(\psi_j(t)) - T(\rho)(\psi_j(t))| < \frac{1}{9}$ for all $t \in \bigcup_k x_{r_k} \setminus x_{r_{k_0}}$.

Now let us assume that there is a $\bar{q} \in P_{\rho, d\cup a}$ extending $\rho \restriction (d \cup a)$, and an $m \in \omega$ such that $\bar{q}$ forces that $\dot{Y}$ contains $(a \setminus m) \cap A_j \cap T(\rho)^{-1}(\frac{2}{3}, \infty)$ and is disjoint from $(a \setminus m) \cap A_j \cap T(\rho)^{-1}(-\infty, \frac{1}{3})$. By enlarging k_0, we can assume that $\psi_j(t) > m$ for all $t \in \bigcup_k x_{r_k} \setminus x_{r_{k_0}}$. Therefore we have that $\bar{q}$ forces that $\psi_j(t) \in \dot{Y}$ for all $t \in \bigcup_k x^+_{r_k} \setminus x_{r_{k_0}}$, and that $\psi_j(t) \notin \dot{Y}$ for all $t \in \bigcup_k x^-_{r_k} \setminus x_{r_{k_0}}$.

Set $q = \bar{q} \restriction (\mathbb{N} \setminus a)$ and notice that $q \in Q$ and so there is an ℓ with $q_\ell = q$. Choose any $k > \ell, k_0$. By symmetry, since q_ℓ is not j-decisive, we may assume there is $t \in x^+_{r_{k+1},j} \setminus x_{r_k,j}$ and an ℓ' such that $q_{\ell'} \cup r_{k+1} \Vdash_Q \psi_j(t) \notin \dot{Y}$. However, since $\mathrm{dom}(\rho \setminus r_{k+1}) \subset a$, we have that $q_{\ell'} \cup \rho < \rho$ is in the poset $P_{\rho,d\cup a} = P_{h+\rho\cdot 1_a,d\cup a}$ and so, by the assumption on $\bar{q}$, forces that $\psi_j(t) \in \dot{Y}$. By our assumption on the name $\dot{Y}$, there is a condition $r \in P_{h,d}$ such that $(\psi_j(t), r) \in \dot{Y}$ and is such that $r \cup q_{\ell'} \cup \rho$ is an extension of $q_{\ell'} \cup \rho$. Of course then, $r \cup q_{\ell'} \cup r_{k+1}$ forces that $\psi_j(t) \in \dot{Y}$ which contradicts that $q_{\ell'} \cup r_{k+1} \Vdash_Q \psi_j(t) \notin \dot{Y}$.

Next we use the Claim to show that L is not a bound on the norm of T. The key idea is that being j-decisive is decidable and so we can build suitably long ψ_j-chains in A_j and then branch away into $5L$ many incomparable extensions that share an element $\psi_j(t)$ forced to be in $\dot{Y}$.

Claim 8. There is a doubly-indexed set $\{g^k_i : i \leq 5L, k \in \omega\} \subset Q$ and an increasing sequence $\{j_k : k \in \omega\} \subset J_a$ such that, for each k and $i \leq 5L$

1. $p_0 \subset g^k_i \subset g^{k+1}_i$,
2. $\mathrm{dom}(g^k_i \setminus p_0) \subset a$,
3. for each $\ell < k$, there is an ℓ' such that $q_\ell \subset q_{\ell'}$ and $q_{\ell'} \cup g^{k+1}_i$ is j_k-decisive,
4. $g^{k+1}_i \restriction (a \cap A_{j_k}) \subset g^{k+1}_{i+1} \restriction (a \cap A_{j_k})$ for $i < 5L$,
5. there is a $t_k \in x^+_{g^{k+1}_{5L},j_k} \cap x^+_{g^{k+2}_i,j_k} \setminus x_{g^{k+1}_i,j_k}$,
6. for all $j \in J_a \cap j_k$ and $i \neq \ell \leq 5L$, $x_{g^{k+2}_i,j} \cup x_{g^{k+2}_\ell,j}$ is not a chain.

Proof of Claim 8. We begin with $g^0_i = p_0$ for each $i \leq 5L$ and $j_{-1} = 0$. Assume that we have selected j_{k-1} and $\{g^k_i : i \leq 5L\}$ for some k. Set $\ell_0 = k$. Choose any $j_k > j_{k-1}$ in J_a so that $\mathrm{dom}(g^k_i) \cap a \cap A_{j_k}$ is empty for all $i \leq 5L$. Choose any extension $\bar{g}^{k+1}_0$ of $g^k_0 \cup (g^k_{5L} \restriction (a \cap A_{j_{k-1}}))$ which is j_k-decisive. Suppose $i < 5L$ and we have chosen $\bar{g}^{k+1}_i$ and a value ℓ_{i+1} so that for each $\ell < \ell_i$ there is an $\ell' < \ell_{i+1}$ such that $q_\ell \subset q_{\ell'}$ and $q_{\ell'} \cup \bar{g}^{k+1}_i$ is j_k-decisive. Choose $\bar{g}^{k+1}_{i+1}$ (in ℓ_{i+1} steps) to be any extension of $g^k_{i+1} \cup (g^k_{5L} \restriction (a \cap A_{j_{k-1}})) \cup (\bar{g}^{k+1}_i \restriction (a \cap A_{j_k}))$ so that there is an ℓ_{i+2} such that for all $\ell < \ell_{i+1}$, there is an $\ell' < \ell_{i+2}$ so that $q_\ell \subset q_{\ell'}$ and $q_{\ell'} \cup \bar{g}^{k+1}_{i+1}$ is j_k-decisive. When choosing $\bar{g}^{k+1}_{5L}$ ensure also that there is $t_k \in x^+_{\bar{g}^{k+1}_{5L},j_k}$ which is not in $x_{\bar{g}^{k+1}_i,j_k}$ for any $i < 5L$. Notice that this construction has ensured that $t_{k-1} \in x^+_{g^{k+1}_i,j_{k-1}}$ for each $i \leq 5L$. Finally, choose g^{k+1}_i to be an extension of $\bar{g}^{k+1}_i$ so that $g^{k+1}_i \restriction (a \cap A_{j_k}) = \bar{g}^{k+1}_i \restriction (a \cap A_{j_k})$ and in such a way that for all $j \in J_a \cap j_k$ and all distinct $\ell, i \leq 5L$, $x_{g^{k+1}_i,j} \cup x_{g^{k+1}_\ell,j}$ is not a chain (this last step is a triviality). □

Now, let us consider $g_i = \bigcup_{k\in\omega} g^k_i$ for each $i \leq 5L$. But also, by the additional properties of T, we can choose $a_1 \subset a$ so that for each $j \in J_a$, $a \cap A_j \setminus a_1$ is finite, and so that for all $i < \ell < 5L$, we have that $g_i \cdot g_\ell \cdot 1_{a_1}$ is constantly 0. Then we have that $T(g_i \cdot 1_{a_1})$ is asymptotically equal to $T(g_i \cdot 1_a)$ and $\Sigma_{i<5L}\ g_i \cdot 1_{a_1}$ has norm at most 1. Also, $T(\Sigma_{i<5L}\ g_i \cdot 1_{a_1})$ is asymptotically equal to $T(\Sigma_{i<5L}\ g_i \cdot 1_a)$. By our assumption, we have that there is some q_ℓ which, for each $i \leq 5L$ has decided on the m and forces that for all $\sigma_j(t_k) > m$ which are in $\dot{Y}$, we must have that $T(g_i \cdot 1_a)(\sigma_j(t_k)) > \frac{1}{3}$.

But now if $q_{\bar{\ell}}$ is any extension of q_ℓ, then for each $k > \bar{\ell}$, there is a further extension $q_{\ell'}$ such that, for each $i < 5L$, $q_{\ell'} \cup g^{k+1}_{i,j_k}$ is j_k-decisive. That is, $q_{\ell'} \cup g^{k+1}_{i,j_k}$ forces that $\psi_{j_k}(t_k) \in \dot{Y}$. Therefore, it follows that $T(\Sigma_{i<5L}\ g_i \cdot 1_{a_1})(\psi_{j_k}(t_k))$ is greater than $(5L)(\frac{2}{9})$ for infinitely many k. Which shows that the norm of T is greater than L. $\square$

Lemma 5. *Given f_α, d_α and $\mathcal{D}_\alpha$ as in the inductive construction, there is an $a_\alpha \in \mathcal{I}^+$ which is disjoint from d_α, a pair $f_{\alpha+1}, d_{\alpha+1}$, and a countable family $\mathcal{D}_{\alpha+1}$ such that for each $\beta < \alpha$*

1. *$d_\alpha \cup a_\alpha \subset d_{\alpha+1}$,*
2. *$d_{\alpha+1} \in \mathcal{J}$,*
3. *$f_{\alpha+1} \cdot 1_{d_\alpha} = f_\alpha$, and $f_{\alpha+1} \cdot 1_{d_{\alpha+1}} = f_{\alpha+1}$,*
4. *a_α is T-orthogonal to $c_\alpha = \mathbb{N} \setminus d_{\alpha+1}$,*
5. *$\mathcal{D}_\alpha \subset \mathcal{D}_{\alpha+1}$ and each $D \in \mathcal{D}_{\alpha+1}$ is a predense subset of $P_{f_{\alpha+1}, d_{\alpha+1}}$,*
6. *if G is any $\mathcal{D}_{\alpha+1}$-generic filter on $P_{f_{\alpha+1}, d_{\alpha+1}}$, then $\mathrm{val}_G(\dot{Y}_\beta)$ does not mod finite separate b_α and e_α.*

Proof. As discussed before the previous lemma, there are a_α, c_α and $h_{\alpha,0} \in C_1$ and $d_{\alpha,0} = \mathbb{N} \setminus (a_\alpha \cup c_\alpha)$ so that

1. $d_{\alpha+1} = d_{\alpha,0} \cup a_\alpha \in \mathcal{J}$,
2. $h_{\alpha,0} \cdot 1_{d_\alpha} = f_\alpha$, and $h_{\alpha,0} \cdot 1_{a_\alpha \cup c_\alpha} = 0$
3. for any $\rho_\alpha \in C_1$ with $\rho_\alpha = \rho_\alpha \cdot 1_{a_\alpha}$, each $D \in \mathcal{D}_\alpha$ is predense in $P_{h_{\alpha,0}+\rho_\alpha, d_{\alpha,0}}$,
4. a_α is T-orthogonal to c_α,

We will recursively choose disjoint infinite subsets $a_{\alpha,n}$ of a_α and functions $\rho_{\alpha,n} = \rho_\alpha \cdot 1_{a_{\alpha,n}}$ so as to “handle” $\dot{Y}_n$. However, in doing so we have to take care when defining $a_{\alpha,n+1}$ to ensure that the full ρ_α will not change the fact that $\dot{Y}_n$ was appropriately handled by $\rho_\alpha \restriction a_{\alpha,n}$. Let us again note that regardless of our choice of ρ_α, each member of $\mathcal{D}_\alpha$ will be predense in $P_{h_{\alpha,0}+\rho_\alpha, d_{\alpha+1}}$.

However, in order to make the first step general enough to handle all later steps, we may suppose we have some countable family $\mathcal{E}_{\alpha,0}$ of predense subsets of $P_{h_{\alpha,0}, \mathbb{N}\setminus(a_\alpha \cup c_\alpha)}$, that must be preserved. Fix any $p_0 \in P_{h_{\alpha,0}, \mathbb{N}\setminus(a_\alpha \cup c_\alpha)}$ and any $\dot{Y}_{\beta_0}$ with $\beta_0 < \alpha$.

To begin, apply Lemma 2 to obtain disjoint subsets, $\tilde{a}_{\alpha,0}$ and $c_{\alpha,0}$, of a_α so that $\tilde{a}_{\alpha,0}$ is T-orthogonal to $c_{\alpha,0}$. These may be chosen so that each are in $\mathcal{I}^+$ and are disjoint from $\mathrm{dom}(p_0)$. Apply Lemma 3 to choose $a_{\alpha,0} \subset \tilde{a}_{\alpha,0}$ and a function $h_{\alpha,1} \in C_1$ so that $h_{\alpha,1} \cdot 1_{d_{\alpha,0}} = h_{\alpha,0}$, $h_{\alpha,1} \cdot 1_{a_{\alpha,0} \cup c_{\alpha,0}} = 0$, and, for all $\rho \in C_1$ with $\rho \cdot 1_{a_{\alpha,0} \cup c_{\alpha,0}} = \rho$, we have that each member of $\mathcal{E}_{\alpha,0}$ is predense in $P_{h_{\alpha,1}+\rho, \mathbb{N}\setminus c_\alpha}$. Set $d_{\alpha,1} = d_{\alpha,0} \cup a_\alpha \setminus (a_{\alpha,0} \cup c_{\alpha,0})$.

This gives us the poset $P_{h_{\alpha,1}, d_{\alpha,1}}$ and first we replace p_0 by the unique extension with domain $d_{\alpha,1}$ which agrees with $h_{\alpha,1}$ at all points not in $\mathrm{dom}(p_0)$. Then we apply Lemma 4, and in this way we obtain $\rho_{\alpha,0} \in C_1$ with $\rho_{\alpha,0} \cdot 1_{a_{\alpha,0}} = \rho_{\alpha,0}$, so that $(p_0 + \rho_{\alpha,0}) \restriction (d_{\alpha,1} \cup a_{\alpha,0})$ forces with respect to the poset $P_{h_{\alpha,1}+\rho_{\alpha,0}, d_{\alpha,1} \cup a_{\alpha,0}}$, that $\dot{Y}_{\beta_0}$ does not mod finite split $a_{\alpha,0} \cap T(h_{\alpha,1} + \rho_{\alpha,0})^{-1}(\frac{2}{3}, \infty)$ and $a_{\alpha,0} \cap T(h_{\alpha,1} + \rho_{\alpha,0})^{-1}(-\infty, \frac{1}{3})$.

Let us note that for all $\rho \in C_1$, $T(\rho \cdot 1_{c_{\alpha,0}}) \cdot 1_{a_{\alpha,0}}$ converges to 0. There is a countable set $\mathcal{E}_{\alpha,1} \supset \mathcal{E}_{\alpha,0}$ of predense subsets of $P_{h_{\alpha,1}+\rho_{\alpha,0}, d_{\alpha,1} \cup a_{\alpha,0}}$ with the property that so long as a filter G with $h_{\alpha,0} + \rho_{\alpha,0} \in G$ meets each element of $\mathcal{E}_{\alpha,1}$, it will ensure that $\mathrm{val}_G(\dot{Y}_{\beta_0})$ does not split $a_{\alpha,0} \cap T(h_{\alpha,1} + \rho_{\alpha,0})^{-1}(\frac{2}{3}, \infty)$ and $a_{\alpha,0} \cap T(h_{\alpha,1} + \rho_{\alpha,0})^{-1}(-\infty, \frac{1}{3})$.

We continue by choosing any $p_1 \in P_{h_{\alpha,1}+\rho_{\alpha,0},d_{\alpha,1}\cup a_{\alpha,0}}$ and any $\beta_1 < \alpha$. We will select $\tilde{a}_{\alpha,1}$, $c_{\alpha,1}$, $a_{\alpha,1}$ as subsets of $c_{\alpha,0}$ as we did with $\tilde{a}_{\alpha,0}$, $c_{\alpha,0}$, $a_{\alpha,0}$. We set $d_{\alpha,2} = \mathbb{N}\setminus(a_{\alpha,1}\cup c_{\alpha,1})$ and $h_{\alpha,2}$ as above so that $h_{\alpha,2}\cdot 1_{d_{\alpha,0}\cup a_{\alpha,0}} = (h_{\alpha,1}+\rho_{\alpha,0})\cdot 1_{d_{\alpha,0}\cup a_{\alpha,0}}$.

The recursion continues for ω-many steps and we define $f_{\alpha+1}$ to be the unique function satisfying that $f_{\alpha+1}\cdot 1_{d_\alpha\cup a_\alpha} = f_{\alpha+1}$ and $f_{\alpha+1}\cdot 1_{d_{\alpha,\ell}\cup a_{\alpha,\ell}} = h_{\alpha,\ell}+\rho_{\alpha,\ell}$ for all $\ell\in\omega$. In this recursion, it is easily arranged that $\bigcup_\ell(d_{\alpha,\ell}\cup a_{\alpha,\ell}) = d_\alpha\cup a_\alpha = d_{\alpha+1}$ and let $\rho_\alpha = f_{\alpha+1}\cdot 1_{a_\alpha}$. Additionally, it is easily arranged that for each n and each pair $p\in P_{h_{\alpha,n},d_{\alpha,n}}$, $\beta<\alpha$, there is an $\ell\geq n$ such that at stage ℓ we are considering $p_\ell = p$ and $\beta_\ell = \beta$.

Choose any $q\in P_{f_{\alpha+1},d_{\alpha+1}}$ and $\beta\in\alpha$. Choose any $k\in\omega$ so that the finite set of places where q might disagree with $f_{\alpha+1}$ is contained in $d_{\alpha,k}$. Let $p = q\restriction d_{\alpha,k}$ and choose $\ell > k$ so that at stage ℓ of this construction, we were considering p and $\dot{Y}_\beta$. This means that at stage ℓ, we were working with $q\restriction d_{\alpha,\ell+1}$ and we arranged that $q\restriction(d_{\alpha,\ell+1}\cup a_{\alpha,\ell})$ forced over the poset $P_{h_{\alpha,\ell+1}+\rho_{\alpha,\ell},d_{\alpha,\ell+1}}$ that $\dot{Y}_\beta$ did not mod finite split $a_{\alpha,\ell}\cap T(h_{\alpha,\ell+1})^{-1}(\frac{2}{3},\infty)$ and $a_{\alpha,\ell}\cap T(h_{\alpha,\ell+1})^{-1}(-\infty,\frac{1}{3})$. We set $\mathcal{E}_{\alpha,\ell+1}$ to be a countable family of predense sets that will ensure this continues to hold, and at stage $\ell+1$, we ensured that for all $\rho\in C_1$ such that $\rho\cdot 1_{a_{\alpha,\ell+1}\cup c_{\alpha,\ell+1}} = \rho$, each member of $\mathcal{E}_{\alpha,\ell+1}$ is predense in $P_{h_{\alpha,\ell+1}+\rho,\mathbb{N}\setminus c_\alpha}$. Define $\mathcal{D}_{\alpha+1}$ to be any countable collection of predense subsets of $P_{f_{\alpha+1},d_{\alpha+1}}$ which contains $\mathcal{D}_\alpha$ and $\bigcup_\ell\mathcal{E}_{\alpha,\ell+1}$. Since $T(f_{\alpha+1})\cdot 1_{a_{\alpha,\ell}}$ is asymptotically equal to $T(h_{\alpha,\ell+1})\cdot 1_{a_{\alpha,\ell}}$, we have completed the proof of the Lemma. □

Lemma 6. *Assume that* $\{d_n : n\in\omega\}$ *is an increasing family of members of* $\mathcal{J}$ *and that* $\{h_n : n\in\omega\}\subset C_1$ *has the property that, for each* n, $h_{n+1}\cdot 1_{d_n} = h_n$. *Then, for any countable family* $\mathcal{E}$ *of predense subsets of the poset* $\bigcup_n P_{h_n,d_n}$, *there is a pair* $h\in C_1$ *and* $d\in\mathcal{J}$ *such that* $\bigcup_n P_{h_n,d_n}\subset P_{h,d}$ *and each* $E\in\mathcal{E}$ *is predense in* $P_{h,d}$.

Proof. Similar to Lemma 3. First to choose d we define $d\cap A_n$ for each n. Choose $d\cap A_n$ so that

1. $A_n\cap d_m\subset d$ for each $m<n$,
2. $(A_n\cap d_m)\setminus d$ is finite for all m,
3. $A_n\setminus d$ is infinite.

Naturally we have ensured that $d\in\mathcal{J}$ and that $d_m\setminus d$ is contained in $\bigcup_{n\leq n} A_n\cap d_m\setminus d$, and so is finite. We will define h so that $h\cdot 1_d = h$ and so that $h\cdot 1_{d_n\cap d} = h_n\cdot 1_{d_n\cap d}$ for each n. However, in order to ensure that each $E\in\mathcal{E}$ is still predense in $P_{h,d}$, we will recursively shrink d while preserving that $d_n\setminus d$ is finite for all n. By recursion on k we will choose a finite set L_k disjoint from d_k, and will redefine d to be $d\setminus\bigcup_n L_n$. Let $\{p_k, E_k : k\in\omega\}$ be an enumeration of all pairs from $\bigcup_n P_{h_n,d_n}$ and $\mathcal{E}$.

Suppose we have chosen L_k and we consider the pair p_k, E_k. Choose n_{k+1} large enough so that there is an $e\in E_k$ compatible with p_k and so that both e and p_k are in P_{h_n,d_n} for some $n<n_{k+1}$. In addition, assume that $\mathrm{dom}(p_k)\setminus d_n$ is contained in $\bigcup_{j<n_{k+1}} A_j$. Choose a finite set $L_{k+1}\subset d_{n_{k+1}}\setminus d_k$ so that $\{\ell : (e\cup p_k)(\ell)\neq h_{n_{k+1}}(\ell)\}$ is contained in $d_k\cup L_{k+1}$. It follows that we will have that p_k and e will be compatible in $P_{h,d}$. □

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