TWO RESULTS ON SPECIAL POINTS

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Abstract. We show that there is a nowhere ccc σ-compact space which has a remote point. We show that it is consistent to have non-compact σ-compact separable space X such that every point of the remainder is a limit of a countable discrete subset of non-isolated points of X. This example shows that one cannot prove in ZFC that every locally compact non-compact space has discrete weak P-points.

1. Introduction

A point $p \in \beta X \setminus X$ is a *remote* point of X if p is not the limit of any nowhere dense subset of X. Remote points were introduced by Fine and Gillman [7]. The reals and, with additional set-theoretic assumptions, many ccc spaces, have been shown to have remote points ([14, 4, 2]). In addition, specifically under the continuum hypothesis, non-pseudocompact spaces of weight $\aleph_1$ have remote points [10] and a weak form of the continuum hypothesis is necessary [3].

Many weakenings of the notion of remote have been considered in which the collection of sets that p should be *remote from* is restricted. Thus a point p could be said to be a remote discrete weak P-point of a space X if p is not in the closure of any countable discrete subset of X. If X has no isolated points then a remote point is a remote discrete weak P-point. van Mill asks in [15, 10.1] if every σ-compact locally compact space of weight at most 2^ω has a remote discrete weak P-point. We show in this paper that this is not the case. The author has asked [6] if there is, in ZFC, a nowhere ccc σ-compact space which has a remote point and we show that this is the case.

Our primary interest will be in spaces which have the form $\Sigma_n X$, i.e. has the form $\omega \times X$, for a compact space X. We will also consider countable free unions of posets P. In each case, we will refer to the elements of $\Sigma_n X$ (or $\Sigma_n P$) using ordered pairs (n, x) (or (n, p)) but when there is no danger of confusion, we will suppress the first coordinate.

We recall the following two results from the Handbook of Boolean algebra [8, 4.11,4.16]

Definition 1.1. (and LEMMA) Let P be a partial order, and for $p \in P$, let $u_p = \{q \in P : q \leq p\}$. The set $\{u_p : p \in P\}$ is a base of the partial order topology on P. A subset u of P is open iff $[p] \in u$ and $q \leq p$ imply $q \in u$.

Any space P with a topology, has an associated complete Boolean algebra, $RO(P)$, consisting of the regular open subsets (join and complement involve taking

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1991 *Mathematics Subject Classification.* 54G05.
Key words and phrases. discrete weak P-points, remote points.
Sponsored by NSF #2975010131; thanks to an excellent referee for the simpler proof of 2.5 .

interior of closures). Although a given u_p need not be regular open, the mapping $e(p) = \text{int cl } u_p$ is an order preserving embedding of $(P, <)$ into $(RO(P), \subset)$.

Proposition 1.2. *Let P be a partial order and $(e, B = RO(P))$ the completion of P constructed in 1.1. The following are equivalent:*

1. *P is separative,*
2. *$e(p) = \text{int cl } u_p$ coincides with u_p, for every $p \in P$,*
3. *e is an isomorphism from P onto the partial order $e[P] \subset B$.*

For each of our two constructions, we will obtain our space by constructing (or examining) a poset. By Proposition 1.2, we may define $B(P)$ to be the Boolean subalgebra of $RO(P)$ which is generated by P. Thus it will be convenient to translate the existence of a remote point on $\Sigma_n\left(S(B(P))\right)$ to a combinatorial property of $\Sigma_n P$. For convenience, if we say that $A = \Sigma_n A_n$ is a subset of $\Sigma_n P$, we will infer that $A \cap (\{n\} \times P_n) = \{n\} \times A_n$.

Definition 1.3. A collection $\mathcal{L}$ of subsets of $\Sigma_n P$ will be a remote filter if

1. for each $L = \Sigma_n L_n \in \mathcal{L}$, L_n is finite for each n,
2. for each maximal antichain $A = \Sigma_n A_n$, there is an $L = \Sigma_n L_n \in \mathcal{L}$ such that $L \subset A$,
3. for any finite set $\mathcal{L}' \subset \mathcal{L}$, there is an n and a $p \in P$ so that for each $\Sigma_n L_n \in \mathcal{L}'$, there is a $q \in L_n$ such that $p < q$.

Lemma 1.4. *Let P be an atomless poset. The space $\Sigma_n S(B(P))$ has a remote point if and only if $\Sigma_n P$ has a remote filter.*

Proof. Let X denote the space $\Sigma_n S(B(P)) = \omega \times S(B(P))$. The assumption that P is atomless is equivalent to the condition that $S(B(P))$ has no isolated points, hence every point of x is in the closure of a nowhere dense subset of X, and so is not a remote point of X. Suppose that $x \in \beta X \setminus X$ is a remote point of X. For each $A = \Sigma_n A_n \subset \Sigma_n P$ which is a maximal antichain, set $U_A = \bigcup\{u_p : p \in A\}$. Since $\{u_p : p \in P\}$ is a dense subset of the Boolean algebra $RO(P)$ and A is a maximal antichain of $\Sigma_n P$, it follows that U_A is a dense open subset of $\Sigma_n S(B(P))$. Since x is a remote point, there is a compact neighborhood K_A of x in βX, such that K_A avoids $X \setminus U_A$. For each n, $K_A \cap (\{n\} \times S(B(P)))$ is a compact subset and so is covered by some finite subset of $\{u_p : p \in A_n\}$. Thus, there is a finite $L_n \subset A_n$ so that x is in the interior of the closure of $\Sigma_n\{u_p : p \in L_n\}$, i.e. this union is a dense open subset of a neighborhood of x. Clearly by the construction, the set $\mathcal{L}$ of families $\Sigma_n L_n$ constructed in this way will satisfy the first and second clause of Definition 1.3. Verification of the third clause follows immediately from the fact that any finite intersection of open sets each of which is dense in a neighborhood of x will again be dense in a neighborhood of x.

For the other direction, assume that $\mathcal{L}$ is a remote filter on $\Sigma_n P$. We basically reverse the steps above. For each $L = \Sigma_n L_n$ in $\mathcal{L}$, let $U_L = \Sigma_n\left(\bigcup\{u_p : p \in L_n\}\right)$. By condition three of Definition 1.3, the collection $\{U_L : L \in \mathcal{L}\}$ will form a filter of open subsets of X. Fix any point x of βX which is in the closure of U_L for all $L \in \mathcal{L}$. We check that x is a remote point of X. Fix any nowhere dense set $D \subset X$. Let $A = \Sigma_n A_n$ be any maximal antichain of $\Sigma_n P$ with respect to the property that $(\text{cl } u_p) \cap D$ is empty for each $p \in A$. Since P is atomless and is dense in $B(P)$, it follows that A is a maximal antichain of $\Sigma_n P$. Let $L = \Sigma_n L_n \in \mathcal{L}$ be any element satisfying the second condition of 1.3, then we have a closed set

$F = \Sigma_n \mathrm{cl}\,(\bigcup\{u_p : p \in L_n\})$ of X which has x in its closure and which is disjoint from the closed set D. Since X is normal, F is completely separated from D, hence F (and x) has a neighborhood in βX which is disjoint from D. □

2. A nowhere ccc space with a remote point

In this section we prove that $\Sigma_n P$ has a remote filter where P is a poset invented by Baumgartner (see Definition 2.1) to illustrate the difference between Axiom A forcings and proper forcings. It is interesting to us because it is nowhere ccc but everywhere ω_1. The combination of its being proper (with finite conditions) and having cardinality ω_1 allow us to generalize an older proof that every compact ccc space with π-weight ω_1 has remote points.

Definition 2.1. [12, VII 4.3A] A condition p is a member of P if p is finite and there is a continuous increasing $f : \omega_1 \to \omega_1$ such that $p \subset f$. This is the same as f being the enumeration of a cub subset of ω_1. P is ordered by simple reverse inclusion: $p < q$ if $p \supset q$.

Let $p \in P$ and let $C \subset \omega_1$ be any cub such that p is a subset of the enumerating function for C. Fix $\delta \in C$ such that $\mathrm{dom}(p) \subset \delta$ and $C \cap \delta$ has order type δ. It is easily shown that $\{p \cup \{(\delta + 1, \gamma)\} : \delta < \gamma < \omega_1\}$ is an uncountable antichain of conditions below p. Therefore, P (and $S(B(P))$) is nowhere ccc.

As we said, Baumgartner shows (see 2.3) that this poset P is proper in a very strong sense. For the rest of the paper, we may fix a regular cardinal θ which is larger than $2^{\aleph_1}$ and let $\mathbb{H}$ denote any sufficiently large submodel of the set-theoretic universe with the property that $\mathbb{H}^{\omega_1} \subset \mathbb{H}$ such as $H(\theta)$ or V_θ. We will use the notion of a subset M of $\mathbb{H}$ being an elementary submodel, denoted $M \prec \mathbb{H}$ (see [9, 11]). For our purposes it should be enough to realize that this means intuitively that if $m_1, \cdots, m_n$ are elements of M, and $\varphi(x_1, \ldots, x_n)$ is a formula of set-theory (using only ϵ and $=$) with all free variables shown, then $\varphi(m_1, \ldots, m_n)$ holds in M if and only if it holds in $\mathbb{H}$.

Definition 2.2. Let P be a poset. A set $D \subset P$ is predense below an element $p \in P$ if for each $q \leq p$, there is a $d \in D$ and an $r \leq q$ such that $r \leq d$. A set is said to be predense if it is predense below every element of P.

Proposition 2.3. *If $P \in M \prec \mathbb{H}$ and $p \in P \cap M$, then $p \cup \{\langle\delta, \delta\rangle\}$ is (P, M)-generic where $\delta = M \cap \omega_1$. That is, if $A \in M$ is any predense subset of P, the subset $A \cap M$ is predense below $p \cup \{\langle\delta, \delta\rangle\}$.*

The above statement implicitly recalls the definition of proper (see [12, III 1.9 and 2.8]) as it applies to P.

Proof. Let $p \in P \cap M$, $\delta = M \cap \omega_1$, and $A \in M$ be predense. To check that $A \cap M$ is predense below $p' = p \cup \{(\delta, \delta)\}$, we let $q \leq p$. By the definition of P, it is clear that $q \cap M = q \cap (\delta \times \delta)$ is also a member of $M \cap P$. Applying elementarity to $q \cap M$ and A, there is an $a \in A \cap M$ such that $a \cup (q \cap M)$ is a member of P. We finish by checking that $a \cup q$ is also a member of P. Fix any cub $C_q \subset \omega_1$ such that q is a subset of the enumeration function of C_q. Also, fix such a cub C' for $a \cup (q \cap M)$ but choose $C' \in M$. Since C' is closed and unbounded, and $M \prec \mathbb{H}$ it follows that $\delta \in C'$. Set $C = (C' \cap \delta) \cup (C_q \setminus \delta)$ and let f_C denote the enumerating function of C. It is easily seen that C is closed, that $(a \cup (q \restriction \delta)) \subset f_C \restriction \delta$, and that $q \restriction [\delta, \omega_1) \subset f_C$. Therefore, $a \cup q$ is in P as required. □

It is useful to make note of the following result which follows directly from the proof.

Corollary 2.4. *If $M \prec \mathbb{H}$ is countable with $P \in M$ and $M \cap \omega_1 = \delta$, then each $p \in P$ such that $p(\delta) = \delta$ is (P, M)-generic.*

If $\mathcal{U}$ is a filter on ω, then the ordering, $<_{\mathcal{U}}$, on ${}^{\omega}\omega$ is given by $f <_{\mathcal{U}} g$ if $\{n : f(n) < g(n)\}$ is a member of $\mathcal{U}$. If $\mathcal{U}$ is an ultrafilter, than $<_{\mathcal{U}}$ determines a linear ordering (on the equivalence classes). We will hereafter fix an ultrafilter $\mathcal{U}$ on ω and let $\kappa_{\mathcal{U}}$ denote the minimum cardinality of a cofinal sequence in $({}^{\omega}\omega, <_{\mathcal{U}})$ (i.e. the ultrapower ordering). It is easily seen that $\kappa_{\mathcal{U}}$ is a regular uncountable cardinal.

Theorem 2.5. *There is a remote filter on $\Sigma_n P$, hence the space $X = \Sigma_n St(RO(P))$ is σ-compact, nowhere ccc and has remote points.*

Proof. Fix any ultrafilter $\mathcal{U}$ on ω and let $\kappa = \kappa_{\mathcal{U}}$. Also, fix any sequence $\{f_\gamma : \gamma < \kappa\}$ which is $\mathcal{U}$-increasing and cofinal in $({}^{\omega}\omega, <_{\mathcal{U}})$.

Throughout the proof, fix for each $\delta \in \omega_1$ a 1-1 enumeration $P_\delta = \{p(\delta, k) : k \in \omega\}$ where $p \in P_\delta$ so long as $p \subset \delta \times \delta$. In effect, we have a function $\iota : \omega_1 \times \omega \to P$, and we will let $\iota_\delta(p) = k$ abbreviate that $p \in P_\delta$ and $p(\delta, k) = p$.

Fix any maximal antichain $A = \Sigma_n A_n$ of $\Sigma_n P$, we show how to construct an $L = \Sigma_n L_n$ to put in a remote collection $\mathcal{L}$. Fix a countable elementary submodel $M_0 = M^A$ of $\mathbb{H}$ so that $\iota, P, \Sigma_n A_n, \mathcal{U}, \{f_\gamma : \gamma < \kappa\}$ are all in M_0. Next, for $0 < j < \omega$, let $M_j \supset M_{j-1}$ be a countable elementary submodel containing $\iota, P, \Sigma_n A_n, \mathcal{U}, \{f_\gamma : \gamma < \kappa\}$ and M_{j-1}. Since M_{j-1} is countable, there is an ordinal $\gamma < \kappa$ such that f_γ is bigger than any function $g \in {}^{\omega}\omega \cap M_{j-1}$ in the ordering $<_{\mathcal{U}}$. By elementary, there is some $\gamma_j^A < \kappa$ in M_j such that $g <_{\mathcal{U}} f_{\gamma_j^A}$ whenever $g \in M_{j-1} \cap {}^{\omega}\omega$.

Let $\delta^A = M^A \cap \omega_1$ and put $\eta^A = \sup\{\gamma_j^A : j \in \omega\}$.

The definition of the required $L = \Sigma_n L_n$ is completely trivial, simply

$$L_n = \{p \in A_n : \iota_{\delta^A}(p) < f_{\eta^A}(n)\} \ .$$

This definition certainly guarantees that $\mathcal{L}$ satisfies the conditions (1) and (2) of 1.3 but we have to check the non-trivial third (filter) condition.

Suppose that $A^i = \Sigma_n A_n^i$ are dense sequences for $i \leq m$ and are enumerated so that $\eta_0 \leq \eta_1 \leq \cdots \leq \eta_m$, where $\eta_i = \eta^{A_i}$. Let us denote similarly $\delta_i = \delta^{A^i}$, $\gamma_j^i = \gamma_j^{A_i}$ and $M_j^i = M_j^{A_i}$. We will let M^i denote M_0^i, hence $\delta_i = M^i \cap \omega_1$.

Let r denote the identity function with domain $\{\delta_i : i \leq m\}$. Clearly $r \in P$ since the set ω_1 is cub in ω_1. Note also that each extension of r is (M, P)-generic for any countable $M \prec \mathbb{H}$ such that $M \cap \omega_1 \in \operatorname{dom} r$, in particular, for every M^i, $i \leq m$.

For each $i \leq m$ choose $\gamma_{j_i}^i$ from the strictly increasing sequence used to define η_i so that $1 < j_i$ and $\gamma_{j_i+1}^i < \gamma_{j_{i+1}-1}^{i+1}$.

Now we prove the following condition $(*)_m$ by induction on m; it is clear that it completes the proof that $\mathcal{L}$ is a remote filter.

$(*)_m$ There is a set $U \in \mathcal{U}$ such that for each $n \in U$ and $i \leq m$, there are $k_i(n) < f_{\gamma_{j_i}^i}(n)$ such that

$$p(\delta_i, k_i(n)) \in A_n^i \quad \text{and} \quad \left(r \cup \bigcup_{i \leq m} p(\delta_i, k_i(n)) \right) \in P \ .$$

If $m = 0$, then for $s = r \restriction \delta_0 + 1$, and $n \in \omega$, there is some $p(n) \in A_n^0 \cap M^0$ such that $s \cup p(n) \in P$ by 2.4. We can find such a $p(n)$ with $\iota_{\delta_0}(p(n))$ minimal.

Then the function $g(n) = \iota_{\delta_0}(p(n)) + 1$ is defined with all parameters in M^0_1. By the definition of $\gamma^0_{j_0}$ we know that the inequality $g(n) < f_{\gamma^0_{j_0}}(n)$ holds for all $n \in U$ for some $U \in \mathcal{U}$. We obviously have that $r \cup p(n)$ is an extension of $r \restriction \delta_0 + 1 \cup p(n)$ which is in P. Therefore setting $k_0(n) = \iota_{\delta_0}(p(n))$ for each n demonstrates the validity of $(*)_0$.

Induction step. Put $s = r \restriction \delta_m$ and define $h : m+1 \to m+1$ by $h(i) = i$ if $\delta_i \leq m$, and $h(i) = m$ otherwise. By induction assumption, there is some $U \in \mathcal{U}$ such that for each $n \in U$ and $i < m$ there is a $k_i(n) < f_{\gamma^i_{j_i}}(n)$ such that $p(\delta_i, k_i(n)) \in A^i_n \cap P_{\delta_i}$ and $r \cup \bigcup_{i<m} p(\delta_i, k_i(n)) \in P$. If we put $p'_{i,n} = p(\delta_i, k_i(n)) \restriction \delta_m$, then we have $r \cup \bigcup_{i<m} p'_{i,n} \in P_{\delta_m}$ as well.

Let $\ell_i(n) \in \omega$ be such that $p'_{i,n} = p(\delta_{h(i)}, \ell_i(n))$. Obviously $\ell_i(n) = k_i(n)$ if $\delta_i \leq \delta_m$. We claim that there is a set $U' \in \mathcal{U}$ such that for each $n \in U'$ and $i < m$, $\ell_i(n) < f_{\gamma^i_{j_i+1}}(n)$. This is a clear consequence of the inductive assumption if $\delta_i \leq \delta_m$, since then $\ell_i(n) = k_i(n)$ and the inequality $k_i(n) < f_{\gamma^i_{j_i}}(n)$ is satisified for all $n \in U$. But if $\delta_m < \delta_i$, we can still capture enough of this relationship between the functions k and ℓ. Define a mapping $g : \omega \to \omega$ by the rule

$$g(n) = \min\{c \in \omega : \text{if } k < f_{\gamma^i_{j_i}}(n) \text{ and } p(\delta_i, k) \restriction \delta_m = p(\delta_m, \ell)\},\ \text{then } \ell < c\}\ .$$

Since all parameters in this formula belong to M^i_j, the mapping g is in M^i_j. Since $g <_{\mathcal{U}} f_{\gamma^i_{j_i}+1}$, there is a set $U_i \in \mathcal{U}$ such that $g(n) < f_{\gamma^i_{j_i}}(n)$ for all $n \in U_i$. It remains to put $U' = U \cap \bigcap_{i<m, \delta_m<\delta_i} U_i$. Note that $k_i(n) < f_{\gamma^i_{j_i}}(n)$ for all $n \in U'$ and so $\ell_i(n) < g(n) < f_{\gamma^i_{j+1}}(n)$.

Let $U'' \subset U'$ be a member of $\mathcal{U}$ such that for all $n \in U''$ and all $i < m$, $f_{\gamma^i_{j_i+1}}(n) < f_{\gamma^m_{j_{m-1}}}(n) < f_{\gamma^m_{j_m}}(n)$.

Now recall that s in (M^m, P)-generic since $s(\delta_m) = \delta_m$. Therefore, we can define a mapping $f : \omega \to \omega$ by the rule: $f(n)$ is the minimal $c < \omega$ satisfying, whenever a finite sequence of integers $\langle \ell_i(n) : i < m \rangle \in f_{\gamma^m_{j_m-1}}(n)^m$ is such that

$$\left(s \cup \bigcup_{i<m} p(\delta_{h(i)}, \ell_i(n)) \right) \in P\ ,$$

then there is an integer $k_m(n) < c$ such that $p(\delta_m, k_m(n)) \in A^m_n$ and

$$p(\delta_m, k_m(n)) \cup \left(s \cup \bigcup_{i<m} p(\delta_{h(i)}, \ell_i(n)) \right) \in P\ .$$

All parameters again belong to $M^m_{j_m-1}$, hence f belongs to $M^m_{j_m-1}$. Since $f <_{\mathcal{U}} f_{\gamma^m_{j_m}}$, there is some $U''' \in \mathcal{U}$, $U''' \subset U''$ such that for all $n \in U'''$, $f(n) < f_{\gamma^m_{j_m}}(n)$.

From $(*)_{m-1}$ and from the fact that s is (M^m, P)-generic, we conclude that for all $n \in U'''$ and $i \leq m$, there is some $k_i(n) < f_{\gamma^i_{j_i}}(n)$ with $p(\delta_i, k_i(n)) \in A^i_n$ such that

$$r \cup \bigcup_{i<m} p(\delta_i, k_i(n)) \in P$$

and with

$$p(\delta_m, k_m(n)) \cup s \cup \bigcup_{i<m} p(\delta_{h(i)}, \ell_i(n)) \in P\ .$$

Since $p(\delta_m, k_m(n)) \in P_{\delta_m}$ and since $p(\delta_{h(i)}, \ell_i(n))$ either equals to $p(\delta_i, k_i(n))$ or is a restriction of some $p(\delta_i, k_i(n))$ to δ_m, we obtain also that

$$r \cup \bigcup_{i \leq m} p(\delta_i, k_i(n)) \in P$$

whenever $n \in U'''$, which shows $(*)_m$. □

3. A separable space with no remote weak P-point

In this section we prove that the space constructed in [5] provides an example which will prove the following theorem.

Theorem 3.1. *It is consistent that there is a compact separable space X with no isolated points such that $\Sigma_n X$ does not have any remote discrete weak P-points. In particular, every point of $\beta(\Sigma_n X)$ is a limit point of a countable nowhere dense discrete subset of $\Sigma_n X$.*

The space X is given as $S(B(P))$ for a poset of the following type.

Definition 3.2. For $Z \subset 2^\omega$, consider $2^{<\omega} \cup Z$ as a subtree of $2^{\leq\omega}$ and define $P_Z = \{a \subset 2^{<\omega} \cup Z : a \text{ is a finite non-maximal antichain}\}$. P_Z is ordered by reverse inclusion.

Since P_Z is separative we can think of the elements of P_Z as corresponding to members of $B(P_Z)$ and also as corresponding to clopen subsets of $S(B(P_Z))$. For $a, b \in P_Z$, we let $a \perp b$ denote the relation that $a \cup b \notin P_Z$ (which means that either $a \cup b$ is maximal or is not an antichain of $2^{<\omega} \cup Z$). Note that being an antichain of P_Z is different than being an antichain of $2^{\leq\omega}$. For $a \in P_Z$, we let $[a]$ denote the set consisting of all branches $b \in 2^\omega$ with the property that either $b \in a$ or $b \restriction n \in a$ for some $n \in \omega$.

P_Z is σ-centered since for each $b \in P_\emptyset$, $\{a \in P_Z : a \cap 2^{<\omega} = b\}$ is centered. It is shown in [5] that if $Z = 2^\omega$ then $\Sigma_N P_Z$ has remote filters. However, the following is also established.

Theorem 3.3. [5] *In the model obtained by adding ω_2 side-by-side Sacks reals, there is an uncountable dense and co-dense subset Z of 2^ω such that $\Sigma_n P_Z$ does not have a remote filter.*

Since every non-remote point is in the closure of a nowhere dense subset, we finish the proof of Theorem 3.1 by establishing the following Lemma.

Lemma 3.4. *If Z is an uncountable dense co-dense subset of 2^ω, every nowhere dense subset of $S(B(P_Z))$ is contained in the closure of a countable discrete subset of $S(B(P_Z))$.*

Proof. Fix any maximal antichain $\{a_n : n \in \omega\}$ of P_Z. Let $\{c_\ell : \ell \in \omega\}$ enumerate $P_\emptyset$. We define $a_{n,m}$ for each n, m so that $\{a_{n,m} : m \in \omega\}$ is an antichain which is predense below a_n and so that for each $\ell \leq n$, and each i, $a_{n,i} \perp c_\ell$ or $c_\ell \subset a_{n,i}$ (it is easy to see that this can be done). We also assume, for convenience, that $a_{n,i} \cap Z$ is not empty for each n, i. Next, for each n, i, we fix any $y_{n,i} \in 2^\omega$ so that $y_{n,i} \notin Z \cup [a_{n,i}]$, hence $a_{n,i} \cup \{y_{n,i}\}$ is an antichain. Since $y_{n,i}$ is not in $[a_{n,i}] \cup Z$, we may let $j_0(n,i)$ be minimal such that $a_{n,i} \cap Z$ is disjoint from $[y_{n,i} \restriction j_0(n,i)]$. Suppose that $b \in P_Z$ is such that $a_{n,i} \subset b$. Clearly $y_{n,i} \notin b$ since $y_{n,i} \notin Z$. In addition, $y_{n,i} \restriction j \notin b$ for each $j < j_0(n,i)$ since, by the minimality of $j_0(n,i)$,

$a_{n,i} \cup \{y_{n,i} \restriction j\}$ is not in P_Z. For each $j \geq j_0(n,i)$, the choice of $y_{n,i}$ and the definition of $j_0(n,i)$ ensure that $a_{n,i,j} = a_{n,i} \cup \{y_{n,i} \restriction j\}$ is an antichain. In addition, since $a_{n,i} \cap Z$ is not empty, $a_{n,i,j} \in P_Z$ since it is not maximal. For $j < j_0(n,i)$, $a_{n,i} \cup \{y_{n,i} \restriction j\}$ is not an antichain, hence it follows that $\{a_{n,i,j} : j_0(n,i) \leq j\}$ is predense below $a_{n,i}$.

Now, for each n, i and $j_0(n,i) \leq j$, we define a filter base on $B(P_Z)$. Let

$$Y_{n,i,j} = \{a' \cup b \in P_Z : a' \subseteq a_{n,i,j} \text{ and } b \subset (Z \setminus [a_{n,i,j}])\} .$$

It can be shown that $Y_{n,i,j}$ is a point (generates an ultrafilter) in $S(B(P_Z))$. We note that the family $\{Y_{n,i,j} : n, i \in \omega, \text{ and } j_0(n,i) \leq j\}$ is discrete. The key properties then are:

(1) $\{a_{n,i} : i \in \omega\}$ is an antichain, $a_n \subset a_{n,i}$ for each i, and for each $\ell \leq n$, c_ℓ is contained in $a_{n,i}$ or it is incompatible with $a_{n,i}$.
(2) for each n, i, $\{a_{n,i,j} : j_0(n,i) \leq j \in \omega\}$ are defined as $a_{n,i} \cup \{y_{n,i} \restriction j\}$ where $y_{n,i} \notin Z$, and, clearly if $j_0(n,i) \leq j_1 < j_2$, then a_{n,i,j_1} and a_{n,i,j_2} are incompatible.

Now suppose that $b \in P_Z$ is such that $b \cup a_n \in P_Z$ for infinitely many n. We will show that there is n, i, j such that $(a_{n,i} \cup b) \in Y_{n,i,j}$. Let ℓ be such that $c_\ell = b \cap 2^{<\omega}$. Fix $n > \ell$ such that $b \cup a_n \in P_Z$. Fix any i such that $b \cup a_{n,i} \in P_Z$, note that $c_\ell \subset a_{n,i}$. Now fix any $j \geq j_0(n,i)$ large enough so that $[y_{n,i} \restriction j]$ is disjoint from $b \cap Z$. Since $b \cap 2^{<\omega} = c_\ell \subset a_{n,i,j}$ and $(b \cap Z) \cap [a_{n,i}]$ is empty, it follows that $(b \cap Z) \cap [a_{n,i} \cup \{y_{n,i} \restriction j\}]$ is also empty. It then follows that $b \in Y_{n,i,j}$ as required.

This proves that the closure of the discrete set $\{Y_{n,i,j} : n, i \in \omega \text{ and } j_0(n,i) \leq j\}$ contains the complement of $\bigcup\{a_n : n \in \omega\}$ (when the latter is consider as an open subset of $S(B(P_Z))$). □

It is an open problem to determine if all extremally disconnected spaces have a discrete weak P-point (also called discretely untouchable). Simon [13] proves if the space is ccc and satisfies $\mathrm{cf}(g(\mathrm{Clop}(X))) > \omega$ then it will have such points, where $g(B)$ for a boolean algebra B is the minimum cardinality of a subfamily which is not contained in a proper complete subalgebra.

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