REVERSIBLE FILTERS

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Abstract. A space is reversible if every continuous bijection of the space onto itself is a homeomorphism. In this paper we study the question of which countable spaces with a unique non-isolated point are reversible. By Stone duality, these spaces correspond to closed subsets in the Čech-Stone compactification of the natural numbers $\beta\omega$. From this, the following natural problem arises: given a space X that is embeddable in $\beta\omega$, is it possible to embed X in such a way that the associated filter of neighborhoods defines a reversible (or non-reversible) space? We give the solution to this problem in some cases. It is especially interesting whether the image of the required embedding is a weak P-set.

1. Introduction

A topological space X is reversible if every time that $f : X \to X$ is a continuous bijection, then f is a homeomorphism. This class of spaces was defined in [10], where some examples of reversible spaces were given. These include compact spaces, Euclidean spaces $\mathbb{R}^n$ (by the Brouwer invariance of domain theorem) and the space $\omega \cup \{p\}$, where p is an ultrafilter, as a subset of $\beta\omega$. This last example is of interest to us.

Given a filter $\mathcal{F} \subset \mathcal{P}(\omega)$, consider the space $\xi(\mathcal{F}) = \omega \cup \{\mathcal{F}\}$, where every point of ω is isolated and every neighborhood of $\mathcal{F}$ is of the form $\{\mathcal{F}\} \cup A$ with $A \in \mathcal{F}$. Spaces of the form $\xi(\mathcal{F})$ have been studied before, for example by García-Ferreira and Uzcátegi ([6] and [7]). When $\mathcal{F}$ is the Fréchet filter, $\xi(\mathcal{F})$ is homeomorphic to a convergent sequence, which is reversible; when $\mathcal{F}$ is an ultrafilter it is easy to prove that $\xi(\mathcal{F})$ is also reversible, as mentioned above. Also, in [2, section 3], the authors of that paper study when $\xi(\mathcal{F})$ is reversible for filters $\mathcal{F}$ that extend to precisely a finite family of ultrafilters (although these results are expressed in a different language).

Let us say that a filter $\mathcal{F} \subset \mathcal{P}(\omega)$ is reversible if the topological space $\xi(\mathcal{F})$ is reversible. It is the objective of this paper to study reversible filters. First, we give some examples of filters that are reversible and others that are non-reversible, besides the trivial ones considered above. Due to Stone duality, every filter $\mathcal{F}$ on ω gives rise to a closed subset $K_{\mathcal{F}} \subset \omega^* = \beta\omega \setminus \omega$ (defined below). Then our main concern is to try to find all possible topological types of $K_{\mathcal{F}}$ when $\mathcal{F}$ is either reversible or non-reversible. Our results are as follows.

- Given any compact space X embeddable in $\beta\omega$, there is a reversible filter $\mathcal{F}$ such that X is homeomorphic to $K_{\mathcal{F}}$. (Theorem 3.2)

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- Given any compact, extremally disconnnected space X embeddable in $\beta\omega$, there is a non-reversible filter $\mathcal{F}$ such that X is homeomorphic to $K_{\mathcal{F}}$. (Theorem 3.5)
- If X is a compact, extremally disconnnected space that can be embedded in ω^* as a weak P-set and X has a proper clopen subspace homeomorphic to itself, then there is a non-reversible filter $\mathcal{F}$ such that X is homeomorphic to $K_{\mathcal{F}}$ and $K_{\mathcal{F}}$ is a weak P-set of ω^*. (Theorem 4.1)
- There is a compact, extremally disconnnected space X that can be embedded in ω^* as a weak P-set and every time $\mathcal{F}$ is a filter with X homeomorphic to $K_{\mathcal{F}}$ and $K_{\mathcal{F}}$ is a weak P-set, then $\mathcal{F}$ is reversible. (Theorem 4.2)
- Given any compact, extremally disconnected space X that is a continuous image of ω^*, there is a reversible filter $\mathcal{F}$ such that X is homeomorphic to $K_{\mathcal{F}}$ and $K_{\mathcal{F}}$ is a weak P-set of ω^*. (Theorem 4.4)

Also, in section 5, using Martin's axiom, we improve some of the results above by constructing filters $\mathcal{F}$ such that $K_{\mathcal{F}}$ is a P-set.

2. Preliminaries and a characterization

Recall that $\beta\omega$ is the Stone space of all ultrafilters on ω and $\omega^* = \beta\omega \setminus \omega$ is the space of free ultrafilters. We will assume the reader's familiarity of most of the facts about $\beta\omega$ from [9]. Recall that a space is an F-space if every cozero set is C^*-embedded. Since ω^* is an F-space we obtain some interesting properties. For example, every closed subset of ω^* of type G_δ is regular closed and every countable subset of ω^* is C^*-embedded. We will also need the more general separation property.

2.1. Theorem [3, 3.3] Let $\mathcal{B}$ and $\mathcal{C}$ be collections of clopen sets of ω^* such that $\mathcal{B}\cup\mathcal{C}$ is pairwise disjoint, $|\mathcal{B}| < \mathfrak{b}$ and $\mathcal{C}$ is countable. Then there exists a non-empty clopen set C such that $\bigcup\mathcal{B} \subset C$ and $(\bigcup\mathcal{C}) \cap C = \emptyset$.

We will be considering spaces embeddable in $\beta\omega$. There is no ZFC characterization of spaces embeddable in $\beta\omega$ but we have the following embedding results. A space is extremally disconnnected (ED, for short) if the closure of every open subset is open.

2.2. Theorem
- [9, 1.4.4] Under CH, any closed suspace of ω^* can be embedded as a nowhere dense P-set.
- [9, 1.4.7] Every compact, 0-dimensional ED space of weight $\leq \mathfrak{c}$ embedds in ω^*.
- [9, 3.5], [4] If X is an ED space and a continuous image of ω^*, then X can be embedded in ω^* as a weak P-set.

Given $A \subset \omega$, we denote $\mathrm{cl}_{\beta\omega}(A) \cap \omega^* = A^*$. Also, if $f : \omega \to \omega$ is any bijection, there is a continuous extension $\beta f : \beta\omega \to \beta\omega$ which is a homeomorphism; denote $f^* = \beta f\restriction_{\omega^*}$.

The Fréchet filter is the filter $\mathcal{F}_r = \{A : \omega \setminus A \in [\omega]^{<\omega}\}$ of all cofinite subsets of ω and we will always assume that our filters extend the Fréchet filter. Each filter $\mathcal{F} \subset \mathcal{P}(\omega)$ defines a closed set $K_{\mathcal{F}} = \{p \in \beta\omega : \mathcal{F} \subset p\}$ that has the property that $A \in \mathcal{F}$ iff $K_{\mathcal{F}} \subset A^*$ and moreover, $K_{\mathcal{F}} = \bigcap\{A^* : A \in \mathcal{F}\}$. Notice that $\xi(\mathcal{F})$ is the quotient space of $\omega \cup K_{\mathcal{F}} \subset \beta\omega$ when $K_{\mathcal{F}}$ is shrunk to a point. The first thing we will do is to find a characterization of reversible filters in terms of continuous maps of $\beta\omega$.

2.3. Lemma Let $\mathcal{F}$ be a filter on ω. Then $\mathcal{F}$ is not reversible if and only if there is a bijection $f : \omega \to \omega$ such that $f^*[K_{\mathcal{F}}]$ is a proper subset of $K_{\mathcal{F}}$.

Proof. First, assume that $g : \xi(\mathcal{F}) \to \xi(\mathcal{F})$ is a continuous bijection that is not open. Then, $g[\omega] = \omega$ so let $f = g{\restriction}_{\omega}\colon \omega \to \omega$, which is a bijection.

Let $A \subset \omega$ such that $K_{\mathcal{F}} \subset A^*$. Then $A \cup \{\mathcal{F}\}$ is open, so by continuity of g we obtain that $g^{\leftarrow}[A \cup \{\mathcal{F}\}] = f^{\leftarrow}[A] \cup \{\mathcal{F}\}$ is also open. Thus, $f^{\leftarrow}[A] \in \mathcal{F}$ which implies that $K_{\mathcal{F}} \subset f^{\leftarrow}[A]^*$. This implies that $f^*[K_{\mathcal{F}}] \subset A^*$. Thus, we obtain that

$$f^*[K_{\mathcal{F}}] \subset \bigcap \{A^* : K_{\mathcal{F}} \subset A^*\} = K_{\mathcal{F}}.$$

Now, since g is not open, there is $B \in \mathcal{F}$ such that $f[B] \notin \mathcal{F}$. Thus, $K_{\mathcal{F}} \not\subset f[B]^*$. Since $f^*[K_{\mathcal{F}}] \subset f[B]^*$, it follows that $K_f \not\subset f^*[K_{\mathcal{F}}]$ so $K_f \neq f^*[K_{\mathcal{F}}]$. We have proved that $f^*[K_{\mathcal{F}}] \subsetneq K_{\mathcal{F}}$.

Now, assume that $f : \omega \to \omega$ is a bijection such that $f^*[K_{\mathcal{F}}] \subsetneq K_{\mathcal{F}}$. Let $g = f \cup \{\langle \mathcal{F}, \mathcal{F} \rangle\}$, let us prove that this function is continuous but not open.

We first prove that g is continuous. Clearly, continuity follows directly for points of ω so let us consider neighborhoods of $\mathcal{F}$ only. Any neighborhood of $\mathcal{F}$ is of the form $A \cup \{\mathcal{F}\}$ with $A \in \mathcal{F}$. Then $K_{\mathcal{F}} \subset A^*$ and $f^*[K_{\mathcal{F}}] \subset A^*$ too, so $K_{\mathcal{F}} \subset f^{\leftarrow}[A]^*$. This implies that $f^{\leftarrow}[A] \in \mathcal{F}$. We obtain that $g^{\leftarrow}[A \cup \{\mathcal{F}\}] = f^{\leftarrow}[A] \cup \{\mathcal{F}\}$ is a neighborhood of $\mathcal{F}$.

Now, let us prove that g is not open. Since $f^*[K_{\mathcal{F}}] \subsetneq K_{\mathcal{F}}$, there exists $B \subset \omega$ such that $f^*[K_{\mathcal{F}}] \subset B^*$ and $K_{\mathcal{F}} \not\subset B^*$. But $K_{\mathcal{F}} \subset f^{\leftarrow}[B]^*$. Then $f^{\leftarrow}[B] \cup \{\mathcal{F}\}$ is a neighborhood of $\mathcal{F}$ with image $g[f^{\leftarrow}[B] \cup \{\mathcal{F}\}] = B \cup \{\mathcal{F}\}$ that is not open. $\square$

So from now on we will always use Lemma 2.3 when we want to check whether a filter is reversible.

According to [10, Section 6], a space is hereditarily reversible if each one of its subspaces is reversible. Given a filter $\mathcal{F}$ on ω, every subspace of $\xi(\mathcal{F})$ is either discrete or of the form $\xi(\mathcal{F}{\restriction}_A)$ for some $A \in [\omega]^{\omega}$. Here $\mathcal{F}{\restriction}_A = \{A \cap B : B \in \mathcal{F}\}$. So call a filter $\mathcal{F}$ hereditarily reversible if $\mathcal{F}{\restriction}_A$ is reversible for all $A \in [\omega]^{\omega}$.

We present some characterizations of properties of $\mathcal{F}$ and their equivalences for $K_{\mathcal{F}}$. The proof of these properties is easy and left to the reader.

2.4. Lemma Let $\mathcal{F}$ be a filter on ω.

(a) $\xi(\mathcal{F})$ is a convergent sequence if and only if $\mathcal{F} = \mathcal{F}_r$ if and only if $K_{\mathcal{F}} = \omega^*$.
(b) $\xi(\mathcal{F})$ contains a convergent sequence if and only if $\mathrm{int}_{\omega^*}(K_{\mathcal{F}}) \neq \emptyset$.
(c) $\xi(\mathcal{F})$ is Fréchet-Urysohn if and only if $K_{\mathcal{F}}$ is a regular closed subset of ω^*.
(d) $\mathcal{F}$ is an ultrafilter if and only if $|K_{\mathcal{F}}| = 1$.
(e) Let $A \subset \omega$. Then $K_{\mathcal{F}{\restriction}_A} = K_{\mathcal{F}} \cap A^*$.

3. First results

From Lemma 2.4, we can easily find all reversible filters that have convergent sequences. Notice that Proposition 3.1 follows from [2, Theorem 2.1]. However, we include a proof to ilustrate a first use of Lemma 2.3.

3.1. Proposition Let $\mathcal{F}$ be a filter on ω such that $\xi(\mathcal{F})$ has a convergent sequence. Then the following are equivalent

(a) $\mathcal{F}$ is the Fréchet filter,
(b) $\mathcal{F}$ is hereditarily reversible, and
(c) $\mathcal{F}$ is reversible.

Proof. From Lemma 2.4 we immediately get that (a) implies (b). That (b) implies (c) is clear so let us prove that (c) implies (a). So assume that $\mathcal{F} \neq \mathcal{F}_r$ has a convergent sequence. By Lemma 2.4 there is $A \subset \omega$ such that $A^* \subset K_{\mathcal{F}}$. And since $\mathcal{F} \neq \mathcal{F}_r$, there is $B \in [\omega]^\omega$ with $\omega \setminus B \in \mathcal{F}$. Thus, $K_f \cap B^* = \emptyset$.

Let $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$ be partitions into two infinite subsets. Now, let $f : \omega \to \omega$ be a bijection such that f is the identity restricted to $\omega \setminus (A \cup B)$, $f[B_1] = B$, $f[B_0] = A_0$ and $f[A] = A_1$. Then it easily follows that $f^*[K_{\mathcal{F}}] = K_{\mathcal{F}} \setminus {A_0}^* \subsetneq K_{\mathcal{F}}$, which shows that $\mathcal{F}$ is not reversible by Lemma 2.3. □

Clearly, every ultrafilter is hereditarily reversible by Lemmas 2.3 and 2.4 (this is known from [10, Example 9]). By considering ultrafilters with different Rudin-Keisler types, we may find many other examples with isolated points. So naturally the question is whether there exists a reversible filter $\mathcal{F}$ that is different from these examples. More precisely, we consider the following formulation of the problem.

> Let X be a space that can be embedded in ω^* and consider a filter $\mathcal{F}$ such that K_f is homeomorphic to X. Is it possible to choose $\mathcal{F}$ in such a way that $\mathcal{F}$ is reversible? or not reversible?

For $X = \omega^*$, both questions have a positive answer. If $\mathcal{F} = \mathcal{F}_r$, then $K_{\mathcal{F}}$ is homeomorphic to ω^* and $\mathcal{F}$ is reversible. Now, say $\omega = A \cup B$ is a partition into infinite subsets and $\mathcal{F} = \{C \subset A : |A \setminus C| = \omega\}$; then $K_{\mathcal{F}}$ is homeomorphic to ω^* and $\mathcal{F}$ is not reversible (Proposition 3.1). In the next result, we shall show that there are many reversible filters that are non-trivial and in fact, any closed subset of ω^* can be realized by one of them.

3.2. Theorem There exists a filter $\mathcal{F}_0$ on ω with the following properties

(a) any filter that extends $\mathcal{F}_0$ is reversible,
(b) $K_{\mathcal{F}_0}$ is crowded and nowhere dense, and
(c) if X is any closed subset of ω^*, there exists a filter $\mathcal{F} \supset \mathcal{F}_0$ such that $K_{\mathcal{F}}$ is homeomorphic to X.

Proof. Let $\{p_n : n < \omega\} \subset \omega^*$ be a sequence of weak P-points with different RK types; that such a collection exists follows from [11]. Let $\omega = \bigcup\{A_n : n < \omega\}$ be a partition into infinite subsets, we may assume that $p_n \in A_n^*$ for all $n < \omega$. Define $\mathcal{F}_0$ to be the filter of all subsets $B \subset \omega$ such that there is $n < \omega$ with $B \cap A_m = \emptyset$, if $m \leq n$; and $B \cap A_m \in p_m$, if $m > n$. It is easy to see that $K_f = \mathrm{cl}_{\omega^*}(\{p_n : n < \omega\}) \setminus \{p_n : n < \omega\}$, notice that this implies that K_f is nowhere dense. Also, since every countable subset of ω^* is $C^\star$-embedded, it follows that K_f is homeomorphic to ω^*. From this, parts (b) and (c) follow.

So we are left to prove part (a). Let $\mathcal{F} \supset \mathcal{F}_0$ be any filter and let $f : \omega \to \omega$ be a bijection such that $f^*[K_{\mathcal{F}}] \subset K_{\mathcal{F}}$, according to Lemma 2.3 we have to prove that

$f^*[K_\mathcal{F}] = K_\mathcal{F}$. Consider the set

$$B = \{n < \omega : f^*(p_n) \in \{p_k : k < \omega\}\}.$$

Notice that $\{p_n : n < \omega\}$ and $\{f^*(p_n) : n \in \omega \setminus B\}$ are disjoint sets of weak P-points of ω^*. Thus, $\{p_n : n < \omega\} \cup \{f^*(p_n) : n \in \omega \setminus B\}$ is a discrete set. But countable sets in an F-space such as ω^* are C^*-embedded so $\mathrm{cl}_{\omega^*}(\{p_n : n < \omega\}) \cap \mathrm{cl}_{\omega^*}(\{f^*(p_n) : n \in \omega \setminus B\}) = \emptyset$. Since $f^*[K_\mathcal{F}] \subset K_\mathcal{F}$, we obtain that $f^*[K_\mathcal{F}] \cap \mathrm{cl}_{\omega^*}(\{f^*(p_n) : n \in \omega \setminus B\}) = \emptyset$. Thus, $K_\mathcal{F} \subset \mathrm{cl}_{\omega^*}(\{p_n : n \in B\})$.

From the fact that the ultrafilters chosen have different RK types, we obtain that $f^*(p_n) = p_n$ for all $n \in B$. From this it follows that in fact, f restricted to $\mathrm{cl}_{\omega^*}(\{p_n : n \in B\})$ is the identity function. Thus, $f^*[K_\mathcal{F}] = K_\mathcal{F}$. □

Next we will produce a non-reversible filter $\mathcal{F}$ with $K_\mathcal{F}$ homeomorphic to any closed subset of X that is ED. First, we will need two lemmas. Notice that an infinite, compact, 0-dimensional and ED space X has weight $\geq \mathfrak{c}$. To see this, consider any pairwise disjoint family $\{U_n : n < \omega\}$ of pairwise disjoint clopen sets and for every $A \subset \omega$, let $V_A = \mathrm{cl}_X(\{U_n : n \in A\})$, which is clopen. Then $\{V_A : A \subset \omega\}$ is a family of $\mathfrak{c}$ different clopen subsets of X.

3.3. Lemma Let $\{X_n : n < \omega\}$ be infinite, compact, 0-dimensional, ED spaces of weight $\mathfrak{c}$. Then there exists a 0-dimensional, ED space Y such that $Y = \bigcup\{Y_n : n < \omega\}$, where

(a) $Y_n \subset Y_{n+1}$ whenever $n < \omega$, and
(b) Y_n is homeomorphic to X_n for each $n < \omega$.

Moreover, Y is normal and has exactly $\mathfrak{c}$ clopen sets.

Proof. Recall that in every 0-dimensional, ED space, all countable subsets are C^*-embedded. Thus, every infinite, compact, 0-dimensional, ED space has a copy of $\beta\omega$. Also, every compact, 0-dimensional, ED space of weight at most $\mathfrak{c}$ can be embedded in ω^*. This implies that for every $n < \omega$, there exists a topological copy of X_n embedded in X_{n+1}.

So for each $n < \omega$, let $e_n : X_n \to X_{n+1}$ an embedding. If $n \leq m < \omega$, denote by $e_n^m : X_n \to X_m$ the composition of all such appropriate embeddings. In the union $\bigcup_{n<\omega}(X_n \times \{n\})$, define an equivalence relation $\langle x, n\rangle \sim \langle y, m\rangle$ and $n \leq m$ if and only if $y = e_n^m(x)$. So let Y be the quotient space under this relation and for each $n < \omega$, let Y_n be the image of $X_n \times \{n\}$ under the corresponding quotient map. It is easy to see that each Y_n is homeomorphic to X_n for each $n < \omega$. Notice that a set U is open in Y if and only if $U \cap Y_n$ is open in Y_n for all $n < \omega$.

First, let us see that Y is normal and 0-dimensional. In fact, we will argue that if F and G are disjoint closed subsets of Y, they can be separated by a clopen subset. For each $n < \omega$, let $F_n = F \cap Y_n$ and $G_n = G \cap Y_n$. Since $F_0 \cap G_0 = \emptyset$ and Y_0 is compact and 0-dimensional, there is a clopen set $U_0 \subset Y_0$ with $F_0 \subset U_0$ and $G_0 \cap U_0 = \emptyset$. Assume that $k < \omega$ and for each $n \leq k$ we have found U_n clopen in Y_n such that if $n \leq k$, then $F_n \subset U_n$, $G_n \cap U_n = \emptyset$ and if $n \leq m < k$ then $U_m \cap Y_n = U_n$. Now, the two sets $F_{k+1} \cup U_k$ and $G_{k+1} \cup (Y_k \setminus U_k)$ are disjoint and closed in Y_{k+1}. Then choose a clopen subset U_{k+1} such that $F_{k+1} \cup U_k \subset U_{k+1}$ and $[G_{k+1} \cup (Y_k \setminus U_k)] \cap U_{k+1} = \emptyset$. This concludes the recursive construction of $\{U_n : n < \omega\}$. Finally, let $U = \bigcup\{U_n : n < \omega\}$, notice that $F \subset U$ and $G \cap U = \emptyset$. Also, U is clopen because $U \cap Y_n = U_n$ is clopen in Y_n for each $n < \omega$.

To see that Y is ED, let $U \subset Y$ be open, we have to prove that $\mathrm{cl}_Y(U)$ is clopen. We will define a sequence of open sets $U_\alpha \subset \mathrm{cl}_Y(U)$ for all ordinals α. Let $U_0 = U$ and if α is a limit ordinal, define $U_\alpha = \bigcup_{\beta<\alpha} U_\beta$. Now assume that U_α is defined and let $U_{\alpha+1} = \bigcup\{\mathrm{cl}_{Y_n}(U_\alpha \cap Y_n) : n < \omega\}$. Since Y_n is closed in Y for every $n < \omega$, $U_{\alpha+1} \subset \mathrm{cl}_Y(U)$. Moreover, Y_n is ED so $\mathrm{cl}_{Y_n}(U_\alpha \cap Y_n)$ is open in Y_n for each $n < \omega$. Also, clearly $\mathrm{cl}_{Y_n}(U_\alpha \cap Y_n) \subset \mathrm{cl}_{Y_m}(U_\alpha \cap Y_m)$ whenever $n < m < \omega$. From this it follows that $U_{\alpha+1}$ is open and we have finished our recursive construction. Notice that $U_\alpha \subset U_\beta$ whenever $\alpha < \beta$. So there exists some $\gamma < |Y|^+$ such that $U_\gamma = U_{\gamma+1}$.

Notice that for all $n < \omega$, $\mathrm{cl}_{Y_n}(U_\gamma \cap Y_n) \subset U_{\gamma+1} \cap Y_n = U_\gamma \cap Y_n$ so in fact $U_\gamma \cap Y_n$ is clopen in Y_n. From this it follows that U_γ is clopen. Since $U \subset U_\gamma \subset \mathrm{cl}_Y(U)$, we obtain that $U_\gamma = \mathrm{cl}_Y(U)$. Then Y is ED.

Since every clopen set U of Y is a union of the clopen subsets $U \cap Y_n$, for $n < \omega$, it follows that there are at most $\mathfrak{c}$ clopen subsets of Y. Also, since Y is normal, Y_0 is C^*-embedded in Y so Y has at least $\mathfrak{c}$ clopen sets. This completes the proof. $\square$

3.4. Lemma Let $\{A_n : n < \omega\}$ be pairwise disjoint infinite subsets of ω and for each $n < \omega$, let K_n be a closed subset of A_n^*. Then $\bigcup\{K_n : n < \omega\}$ is C^*-embedded in $\beta\omega$.

Proof. Let $f : \bigcup\{K_n : n < \omega\} \to [0,1]$ be a continuous function. Given $n < \omega$, since K_n is closed in βA_n, there is a function $g_n : A_n \to [0,1]$ such that $\beta g_n\restriction_{K_n} = f_{K_n}$. So if $g : \omega \to [0,1]$ is any function extending $\bigcup\{g_n : n < \omega\}$, then $\beta g : \beta\omega \to [0,1]$ is an extension of f. $\square$

3.5. Theorem Let X be any compact, 0-dimensional, ED space of weight $\leq \mathfrak{c}$. Then there is a non-reversible filter $\mathcal{F}$ on ω such that $K_{\mathcal{F}}$ is homeomorphic to X.

Proof. Let $\{X_n : n < \omega\}$ be a family of pairwise disjoint clopen subsets of X. Let $B \subset \omega$ with $|B| = |\omega \setminus B| = \omega$, let $\{A_n : n \in \mathbb{Z}\}$ be a partition of $\omega \setminus B$ into infinite subsets and let $f : \omega \to \omega$ be a bijection such that $f\restriction_B$ is the identity function in B and for all $n \in \mathbb{Z}$, $f[A_n] = A_{n+1}$.

By Lemma 3.3, there is an 0-dimensional, ED space Y with exactly $\mathfrak{c}$ clopen sets that is equal to the increasing union of spaces $\{Y_n : n < \omega\}$ such that Y_n is homeomorphic to X_n and $Y_n \subset Y_{n+1}$ for all $n < \omega$. Recall that βY is also ED ([9, 1.2.2(a)]). Also, in a compact and 0-dimensional space the weight is equal to the number of clopen sets so βY has weight $\mathfrak{c}$. Thus, there is an embedding $e : Y \to A_0^*$ ([9, 1.4.7]).

Let $e_0 = e$ and if $n < \omega$, let $e_{n+1} = f^* \circ e_n : Y \to A_{n+1}^*$. For each $n < \omega$, let $Z_n = e_n[X_n]$. Define $Z = \bigcup\{Z_n : n < \omega\}$ and let W be a subset of B^* homeomorphic to the set $X \setminus \mathrm{cl}_X(\bigcup\{X_n : n < \omega\})$. Notice that $\bigcup\{X_n : n < \omega\}$ is a C^*-embedded subset of X because X is ED and $\bigcup\{Z_n : n < \omega\}$ is C^*-embedded in Z by Lemma 3.4. Thus, there is an embedding $h : \mathrm{cl}_X(\bigcup\{X_n : n < \omega\}) \to \omega^*$ such that $h[X_n] = Z_n$ for all $n < \omega$. Since X is extremally disconnected, we may extend h to an embedding $H : X \to \omega^*$ in such a way that $H[X \setminus \mathrm{cl}_X(\bigcup\{X_n : n < \omega\})] = W$.

So let $\mathcal{F}$ be the filter of all $A \subset \omega$ with $Z \cup W \subset A^*$. We will prove that $\mathcal{F}$ is not reversible by showing that $f^*[Z \cup W] \subsetneq Z \cup W$. First, notice that $f^*[W] = W$ and $f^*[Z_n] = e_{n+1}[X_n] \subset Z_{n+1}$ for all $n < \omega$. Finally, $f^*[Z \cup W] \cap A_0^* = \emptyset$ so $f^*[Z \cup W] \cap Z_0 = \emptyset$. This completes the proof. $\square$

4. Embedding as weak P-sets

Recall that every ED space that is a continuous image of ω^* can be embedded in ω^* as a weak P-set ([9, 3.5], [4]). So now we study a problem similar to the one in the previous section, adding the requirement that the embedded space is a weak P-set. More carefully stated, we want the following.

> Let X be a space that can be embedded in ω^* as a weak P-set and consider a filter $\mathcal{F}$ such that K_f is a weak P-set homeomorphic to X. Is it possible to choose $\mathcal{F}$ in such a way that $\mathcal{F}$ is reversible? or not reversible?

First, we start finding filters that are not reversible. The construction is similar to that in Theorem 3.5. However, it needs an extra hypothesis.

4.1. Theorem Let X be a compact ED space that can be embedded in ω^* as a weak P-set. Moreover, assume that there exists a proper clopen subspace of X homeomorphic to X. Then there is a non-reversible filter $\mathcal{F}$ on ω such that $K_{\mathcal{F}}$ is a weak P-set homeomorphic to X.

Proof. From the hypothesis on X, it is easy to find a collection of non-empty, pairwise disjoint clopen sets $\{X_n : n < \omega\}$ of X that are pairwise homeomorphic. Let $B \subset \omega$ with $|B| = |\omega \setminus B| = \omega$, let $\{A_n : n \in \mathbb{Z}\}$ be a partition of $\omega \setminus B$ into infinite subsets and let $f : \omega \to \omega$ be a bijection such that $f{\restriction}_B$ is the identity function in B and for all $n \in \mathbb{Z}$, $f[A_n] = A_{n+1}$.

It is not hard to argue that there is an embedding $e : \bigcup\{X_n : n < \omega\} \to \bigcup\{A_n^* : n < \omega\}$ in such a way that for each $n < \omega$, $e[X_n]$ is a weak P-set of A_n^* and $f^*[e[X_n]] = e[X_{n+1}]$. Since $\bigcup\{X_n : n < \omega\}$ is C^*-embedded in X and X is ED, we may assume that $X \subset \omega^*$, e is the identity function and $X \setminus \mathrm{cl}_X(\bigcup\{X_n : n < \omega\})$ is a weak P-set of B^*.

Now let us see that with these conditions, X is in fact a weak P-set. Let $\{x_n : n < \omega\}$ be disjoint from X. Then for each $n < \omega$, X_n is a weak P-set so $\mathrm{cl}_{\omega^*}(\{x_n : n < \omega\}) \cap X_n = \emptyset$. Thus, the family $\{X_n : n < \omega\} \cup \{\{x_n\} : n < \omega\}$ is discrete and countable so it can be separated by pairwise disjoint clopen sets. By Lemma 3.4, it easily follows that $\bigcup\{X_n : n < \omega\}$ can be separated from $\{x_n : n < \omega\}$ by a continuous function. Also, $\mathrm{cl}_{\omega^*}(\{x_n : n < \omega\}) \cap (X \setminus \mathrm{cl}_X(\bigcup\{X_n : n < \omega\})) = \emptyset$. So in fact $\mathrm{cl}_{\omega^*}(\{x_n : n < \omega\}) \cap X = \emptyset$, which is what we wanted to prove.

Finally, let $\mathcal{F}$ be the neighborhood filter of X so that $K_{\mathcal{F}} = X$. It remains to notice that $f^*[X] \subset X \setminus A_0^* \subsetneq X$. Thus, the statement of the theorem follows. □

Next, we would like to show that the extra hypothesis of Theorem 4.1 is really necessary.

4.2. Theorem There exists a compact ED space X that can be embedded in ω^* as a weak P-set and such that every time $\mathcal{F}$ is a filter with $K_{\mathcal{F}}$ a weak P-set homeomorphic to X then $\mathcal{F}$ is reversible.

Proof. In [5] it was shown that there exists a separable, ED, compact space X that is rigid in the sense that the identity function is its only autohomeomorphism. Using very similar arguments, it can be easily proved that no clopen subset of X

is homeomorphic to X. Since X is separable and crowded, it is easy to see that X is a continuous image of ω^*. This in turn implies that X can be embedded in ω^* as a weak P-set.

Assume now that $\mathcal{F}$ is any filter on ω such that $K_{\mathcal{F}}$ is a weak P-set homeomorphic to X. Let $f : \omega \to \omega$ be a bijection such that $f^*[K_{\mathcal{F}}] \subset K_{\mathcal{F}}$ and assume that $U = K_{\mathcal{F}} \setminus f^*[K_{\mathcal{F}}] \neq \emptyset$. Then, since X is separable, there is a countable set $D \subset U$ with $\mathrm{cl}_{\omega^*}(D) = \mathrm{cl}_{K_{\mathcal{F}}}(U)$. Since $D \cap f^*[K_{\mathcal{F}}] = \emptyset$ and $f^*[K_{\mathcal{F}}]$ is a weak P-set, it follows that $\mathrm{cl}_{\omega^*}(D) \cap f^*[K_{\mathcal{F}}] = \emptyset$. Thus, $\mathrm{cl}_{K_{\mathcal{F}}}(U) \cap f^*[K_{\mathcal{F}}] = \emptyset$ which shows that $U = \mathrm{cl}_{K_{\mathcal{F}}}(U)$ and $f^*[K_{\mathcal{F}}]$ is clopen in $K_{\mathcal{F}}$. So $f^*[K_{\mathcal{F}}]$ is a clopen set of $K_{\mathcal{F}}$ homeorphic to itself, which implies $f^*[K_{\mathcal{F}}] = K_{\mathcal{F}}$. This is a contradiction so in fact $U = \emptyset$ and $f^*[K_{\mathcal{F}}] = K_{\mathcal{F}}$. This shows that $\mathcal{F}$ is reversible. □

We finally consider filters that are reversible. In order to make the corresponding spaces weak P-sets of ω^*, we will need to use Kunen's technique of a construction of a weak P-point ([8]). We shall use Dow's approach from [4].

First, let us recall the concept of a $\mathfrak{c}$-OK set. So let κ be an infinite cardinal, X a space and K closed in X. Given an increasing sequence $\{C_n : n < \omega\}$ of closed subsets of X disjoint from K, we will say that K is κ-OK with respect to $\{C_n : n < \omega\}$ if there is a set $\mathfrak{U}$ of neighborhoods of K such that $|\mathfrak{U}| = \kappa$ and every time $0 < n < \omega$ and $\mathfrak{U}_0 \in [\mathfrak{U}]^n$, $\bigcap \mathfrak{U}_0 \cap C_n = \emptyset$. Then K is κ-OK if it is κ-OK with respect to every countable increasing sequence of closed subsets of X. It easily follows that if a closed set is κ-OK for κ uncountable, then it is a weak P-set (see [8, Lemma 1.3]).

In [4, Lemma 3.2], Dow proves that if ω^* maps onto X, then there is an continuous surjection $\varphi : \omega^* \to X \times (\mathfrak{c}+1)^{\mathfrak{c}}$, where $\mathfrak{c}+1 = \mathfrak{c} \cup \{\mathfrak{c}\}$ is taken as the one-point compactification of the discrete space $\mathfrak{c}$. This map φ will replace Kunen's independent matrices from [8]. Lemma 3.4 in [4] gives a method to construct $\mathfrak{c}$-OK points in ω^* using this map φ. We will use the following modification mentioned by Dow by the end of the proof of [4, Theorem 3.5]. For any set $I \subset \mathfrak{c}$, we denote by $\pi_I : X \times (\mathfrak{c}+1)^{\mathfrak{c}} \to X \times (\mathfrak{c}+1)^I$ the projection. To be consistent with notation, $\pi_\emptyset$ will denote the projection of $X \times (\mathfrak{c}+1)^{\mathfrak{c}} \to X$.

4.3. Lemma Let $\psi : \omega^* \to X \times (\mathfrak{c}+1)^I$ be continuous and onto, where $I \subset \mathfrak{c}$ is an infinite set. Assume that $K \subset \omega^*$ is a closed set with $\psi[K] = X \times (\mathfrak{c}+1)^I$ and $\{C_n : n < \omega\}$ is a sequence of closed subsets of ω^* disjoint from K. Then there is a countable set $J \subset I$ and a closed subset $K' \subset K$ such that

- $(\pi_{I\setminus J} \circ \psi)[K'] = X \times (\mathfrak{c}+1)^{I\setminus J}$, and
- K' is $\mathfrak{c}$-OK with respect to $\{C_n : n < \omega\}$.

Recall in $(\mathfrak{c}+1)^J$, where J any set, there is a base of clopen subsets of the form $\prod\{U_\xi : \xi \in J\}$ where each factor U_ξ is clopen and the support $\{\xi \in J : U_\xi \neq \mathfrak{c}+1\}$ is finite.

4.4. Theorem Let X be a compact ED space that is a continuous image of ω^*. Then there is a reversible filter $\mathcal{F}$ such that $K_{\mathcal{F}}$ is a weak P-set homeomorphic to X.

Proof. Let $\varphi : \omega^* \to X \times (\mathfrak{c}+1)^{\mathfrak{c}}$ be the surjection from [4, Lemma 3.2]. Our objective is to recursively construct a closed set $K \subset \omega^*$ such that $(\pi_\emptyset \circ \varphi)[K] = X$

and $(\pi_\emptyset \circ \varphi)\restriction_K \colon K \to X$ is irreducible. By a classic result by Gleason (see, for example, the argument in [9, 1.4.7]) it follows that $(\pi_\emptyset \circ \varphi)\restriction_K$ is a homeomorphism. So it only remains to take $\mathcal{F}$ to be the filter of neighborhoods of K.

We will define K as the intersection of a family $\{K_\alpha : \alpha < \mathfrak{c}\}$ of closed subsets of ω^*, ordered inversely by inclusion. We will also define a decreasing sequence $\{I_\alpha : \alpha < \mathfrak{c}\} \subset \mathfrak{c}$ such that $I_0 = \mathfrak{c}$ and $|\mathfrak{c} \setminus I_\alpha| < |\alpha| \cdot \omega$ for all $\alpha < \mathfrak{c}$. We will have the following conditions:

(a) If $\beta < \mathfrak{c}$ is a limit, $K_\beta = \bigcap\{K_\alpha : \alpha < \mathfrak{c}\}$ and $I_\beta = \bigcap\{I_\alpha : \alpha < \mathfrak{c}\}$.
(b) For each $\alpha < \mathfrak{c}$, $(\pi_{I_\alpha} \circ \varphi)[K_\alpha] = X \times (\mathfrak{c}+1)^{I_\alpha}$.

We need to do acomplish three things in our construction: make K a weak P-set, that the map $(\pi_\emptyset \circ \varphi)\restriction_K \colon K \to X$ is irreducible and make sure that the filter of neighborhoods $\mathcal{F}$ is reversible. So we will partition ordinals into three sets. For $i \in \{0, 1, 2\}$, let Λ_i be the set of ordinals $\alpha < \mathfrak{c}$ such that $\alpha = \beta + n$, β is a limit ordinal and $n < \omega$ is congruent to i modulo 3. Let $\{\{C_n^\alpha : n < \omega\} : \alpha \in \Lambda_0\}$ be an enumeration of all countable increasing of clopen sets where each sequence is repeated cofinally often. Let $\{B_\alpha : \alpha \in \Lambda_1\}$ be an enumeration of all clopen subsets of ω^*. For these two types of steps we need the following conditions.

(c) Let $\alpha \in \Lambda_0$. If K_α is disjoint from all the members of the sequence $\{C_n^\alpha : n < \omega\}$, then $|I_\alpha \setminus I_{\alpha+1}| \leq \omega$ and $K_{\alpha+1}$ is $\mathfrak{c}$-OK with respect to $\{C_n^\alpha : n < \omega\}$.
(d) Let $\alpha \in \Lambda_1$. If $(\pi_{I_\alpha} \circ \varphi)[K_\alpha \cap B_\alpha] = X \times (\mathfrak{c}+1)^{I_\alpha}$, then $I_{\alpha+1} = I_\alpha$ and $K_{\alpha+1} = K_\alpha \cap B_\alpha$. Otherwise, there are clopen sets $C \subset X$ and $D \subset (\mathfrak{c}+1)^{\mathfrak{c}}$ such that the support of D is equal to $I_\alpha \setminus I_{\alpha+1}$, $\varphi[K_\alpha \cap B_\alpha] \cap (C \times D) = \emptyset$ and $K_{\alpha+1} = K_\alpha \cap \varphi^{\leftarrow}[X \times D]$.

Clearly, (c) follows from Lemma 4.3 and implies that K is a weak P-set of ω^*. Also, it is not hard to see that condition (d) implies that $(\pi_\emptyset \circ \varphi)\restriction_K \colon K \to X$ is irreducible. Conditions (c) and (d) are taken from the proof of [4, Theorem 3.5].

Finally, we need to take care of reversibility using the $\mathfrak{c}$ chances we get from Λ_2. Let $\{f_\alpha : \alpha \in \Lambda_2\}$ be an enumeration of all bijections from ω onto itself, each one repeated cofinally often. We will require the following condition.

(e) Let $\alpha \in \Lambda_2$. Assume that there exists a clopen sets $U \subset \omega^*$ and $V \subset X$ such that $(\pi_{I_\alpha} \circ \varphi)[K_\alpha \cap U] = V \times (\mathfrak{c}+1)^{I_\alpha}$ and $(\pi_\emptyset \circ \varphi)[f_\alpha^*[K_\alpha \cap U]] \subset X \setminus V$. Then $|I_\alpha \setminus I_{\alpha+1}| < \omega$ and there is $x \in X$ such that $f_\alpha^*[K_{\alpha+1} \cap (\pi_\emptyset \circ \varphi)^{\leftarrow}(x)] \cap K_{\alpha+1} = \emptyset$.

Before we show how to prove that (e) can be obtained, let us show why it implies that the filter of neighborhoods of K is reversible. Assume that after our construction, K is not reversible. Then by Lemma 2.3, there is a bijection $f : \omega \to \omega$ such that $f^*[K] \subsetneq K$. By property (d) above we know that $(\pi_\emptyset \circ \varphi)\restriction_K \colon K \to X$ is irreducible so $(\pi_\emptyset \circ \varphi \circ f^*)[K]$ is a proper closed subset of X. Let V be a clopen set disjoint from $(\pi_\emptyset \circ \varphi \circ f^*)[K]$. Now consider the clopen subset $W = (\pi_\emptyset \circ \varphi)^{\leftarrow}[V]$ of ω^*. From the definition of K and the facts that f^* is a homeomorphism and $f^*[K] \cap W = \emptyset$, there is $\beta < \mathfrak{c}$ such that $f^*[K_\gamma] \cap W = \emptyset$ every time $\beta \leq \gamma < \mathfrak{c}$. So fix $\alpha \in \Lambda_2$ such that $\beta \leq \alpha$ and $f_\alpha = f$. Now define

$$U = W \cap ((f^*)^{\leftarrow}[\omega^* \setminus W]),$$

which is a clopen set of ω^* with the property that $(\pi_\emptyset \circ \varphi \circ f^*)[U] \subset X \setminus V$. From the choice of α we obtain that $K_\alpha \cap W \subset U$. Also, $(\pi_{I_\alpha} \circ \varphi)[K_\alpha \cap W] = V \times (\mathfrak{c}+1)^{I_\alpha}$ by property (b). Finally, notice that $K_\alpha \cap W = K_\alpha \cap U$. Thus, our choice of α, U

and V satisfy the hypothesis of condition (e). So let x as in the conclusion of (e) and take $p \in K$ such that $(\pi_\emptyset \circ \varphi)(p) = x$. Then $p \in K_{\alpha+1} \cap (\pi_\emptyset \circ \varphi)^{\leftarrow}(x)$ implies that $f^*_\alpha(p) \notin K_{\alpha+1}$. Thus, $p \in K \setminus f^*[K]$, a contradiction. This contradiction comes from the fact that we assumed that K was not reversible.

So we are left to prove that condition (e) can be achieved. So assume we have α, U and V like in the hypothesis of (e). Choose any $i \in I_\alpha$ and let $J = I_\alpha \setminus \{i\}$. For each $\xi \in \mathfrak{c}$, let $U_\xi = (\pi_{\{i\}} \circ \varphi)^{\leftarrow}(X \times \{\xi\})$, which is a clopen set in X. Then $\{U_\xi : \xi \in \mathfrak{c}\}$ is a pairwise disjoint collection of clopen subsets of X such that $(\pi_J \circ \varphi)[K_\alpha \cap U_\xi] = X \times (\mathfrak{c}+1)^J$ for all $\xi \in \mathfrak{c}$. For each $\xi \in \mathfrak{c}$, consider the set $V_\xi = (f^*_\alpha)^{\leftarrow}[U_\xi] \cap K_\alpha \cap U$, which is a clopen set of $K_\alpha \cap U$. Here we will have two cases.

Case 1: There exists $\xi_0 \in \mathfrak{c}$ such that $(\pi_J \circ \varphi)[V_{\xi_0}] = V \times (\mathfrak{c}+1)^J$. Choose any $\xi_1 \in \mathfrak{c} \setminus \{\xi_0\}$, let $I_{\alpha+1} = J$ and define

$$K_{\alpha+1} = V_{\xi_0} \cup (K_\alpha \cap U_{\xi_1} \cap (\pi_\emptyset \circ \varphi)^{\leftarrow}[X \setminus V]).$$

Notice that $K_{\alpha+1} \subset K_\alpha$ and $(\pi_{I_{\alpha+1}} \circ \varphi)[K_{\alpha+1}] = X \times (\mathfrak{c}+1)^{I_{\alpha+1}}$. Now take any $x \in V$. Then $K_{\alpha+1} \cap (\pi_\emptyset \circ \varphi)^{\leftarrow}(x) \subset V_{\xi_0}$ so $f^*[K_{\alpha+1} \cap (\pi_\emptyset \circ \varphi)^{\leftarrow}(x)] \subset U_{\xi_0}$. Since $U_{\xi_0} \cap U_{\xi_1} = \emptyset$, then $f^*[K_{\alpha+1} \cap (\pi_\emptyset \circ \varphi)^{\leftarrow}(x)] \cap K_{\alpha+1} = \emptyset$ and we are done.

Case 2: Not Case 1. Take $\xi_0 \in \mathfrak{c}$, then there exists clopen sets $C \subset V$ and $D \subset (\mathfrak{c}+1)^J$ such that $C \times D$ is disjoint from $(\pi_J \circ \varphi)[V_{\xi_0}]$. Let $J' \subset J$ be the support of D and define $I_{\alpha+1} = J \setminus J'$. In this case, define

$$K_{\alpha+1} = (K_\alpha \cap U \cap (\pi_J \circ \varphi)^{\leftarrow}[C \times D]) \cup (K_\alpha \cap U_{\xi_0} \cap (\pi_J \circ \varphi)^{\leftarrow}[(X \setminus C) \times D]).$$

Clearly, $K_{\alpha+1} \subset K_\alpha$. It is not hard to see that and $(\pi_J \circ \varphi)[K_{\alpha+1}] = X \times D$, which in turn implies that $(\pi_{I_{\alpha+1}} \circ \varphi)[K_{\alpha+1}] = X \times (\mathfrak{c}+1)^{I_{\alpha+1}}$. Now let $x \in C$. Assume there is $p \in K_{\alpha+1}$ such that $(\pi_\emptyset \circ \varphi)(p) = x$ and $q = f^*_\alpha(p) \in K_{\alpha+1}$, we will reach a contradiction. Notice that $p \in U$, which implies that $q \in (\pi_\emptyset \circ \varphi)^{\leftarrow}[X \setminus V]$. So from the definition of $K_{\alpha+1}$ we obtain that $q \in U_{\xi_0}$. This in turn implies that $p \in V_{\xi_0}$. By the choice of $C \times D$ we obtain that $(\pi_J \circ \varphi)(p) \notin C \times D$. But since $x \in C$, $p \in K_{\alpha+1} \cap U \cap (\pi_J \circ \varphi)^{\leftarrow}[C \times D])$ so $(\pi_J \circ \varphi)(p) \in C \times D$. So we obtain a contradiction and we obtain the negation of our assumption. Thus, $f^*[K_{\alpha+1} \cap (\pi_\emptyset \circ \varphi)^{\leftarrow}(x)] \cap K_{\alpha+1} = \emptyset$, which is what we wanted to prove.

These two cases complete the proof of condition (e) and finish the proof of the Theorem. □

5. Filters generated by towers

It is well known that Martin's axiom (henceforth, **MA**) implies that there are filters that are P-filters (see, for example, [1, Theorem 4.4.5]). Equivalently, there is a filter $\mathcal{F}$ such that $K_{\mathcal{F}}$ is a P-set. It is not hard to see that by changing all instances of "weak P-set" to just "P-set" in Theorem 4.1, we obtain a valid statement. Also, every P-set in a weak P-set so the P-set version of Theorem 4.2 is in fact implied by Theorem 4.2.

As for the P-set version of Theorem 4.4, we will do something stronger, but only for separable, first countable spaces. Recall that a tower is a set $\{A_\alpha : \alpha < \kappa\} \subset \mathcal{P}(\omega)$, for some κ, such that

- $A_\beta \setminus A_\alpha$ is finite every time $\alpha < \beta < \kappa$, and

- there is no $A \subset \omega$ such that $A \setminus A_\alpha$ is finite for every $\alpha < \kappa$.

In this case, $\{A^*_\alpha : \alpha < \kappa\}$ is a decreasing chain of clopen subsets of ω^* with nowhere dense intersection. Clearly, every tower generates a filter and every filter generated by a tower of height $\kappa = \mathfrak{c}$ is a P-filter. In fact, every filter generated by a tower of height $\mathfrak{c}$ is a $P_{\mathfrak{c}}$-filter.

In what follows we will assume the reader's familiarity with **MA** and small uncountable cardinals from [3]. One fact that we will use several times is that **MA** implies that every intersection of less than $\mathfrak{c}$ many clopen sets is a regular closed set (this follows from Theorem 2.1).

5.1. Lemma Let X_0, X_1 be compact separable spaces of weight $< \mathfrak{c}$ and let $\psi_0 : \omega^* \to X_0$ be a continuous onto function. Assume that there is a continuous function $\pi : X_1 \to X_0$ and a partition $X_1 = V_0 \cup V_1$ into two clopen sets such that $\pi\restriction_{V_i} : V_i \to X_0$ is an embedding for $i < 2$. Then **MA** implies there exists a clopen set $W \subset \omega^*$ and a continuous onto function $\psi_1 : W \to X_1$ such that $\pi \circ \psi_1 = \psi_0\restriction_W$.

Proof. Let $F_i = \pi[V_i]$ for $i < 2$, this is a closed subset of X_0. Choose a countable dense set $\{d_n : n < \omega\}$ of X_0 that is contained in the dense open set $(X_0 \setminus F_0) \cup (X_0 \setminus F_1) \cup (\mathrm{int}_{X_0}(F_0 \cap F_1))$. Let $N_i = \{n < \omega : d_n \in X_0 \setminus F_i\}$ for $i < 2$ and $N_2 = \omega \setminus (N_0 \cup N_1)$.

Since F_0 is an intersection of $< \mathfrak{c}$ many clopen sets of X_0, there is a collection $\mathcal{G}_0$ of clopen sets of ω^* such that $\bigcap \mathcal{G}_0 = \psi_0^{\leftarrow}[F_0]$ and $|\mathcal{G}_0| < \mathfrak{c}$. For each $n \in N_0 \cup N_2$, let U_n be a clopen set of ω^* such that $\psi_0[U_n] = \{d_n\}$. Clearly, $U_n \subset \bigcap \mathcal{G}_0$ for all $n \in N_0 \cup N_2$. Thus, considering the collection $\mathcal{G}_0 \cup \{U_n : n \in N_0 \cup N_2\}$, by **MA** and Lemma 2.1, there exists an clopen set $W_0 \subset \omega^*$ such that $W_0 \subset \bigcap \mathcal{G}_0$ and $U_n \subset W_0$ for all $n \in N_0 \cup N_2$. It follows that W_0 is a clopen set of ω^* such that $d_n \in \psi[W_0]$ for all $n \in N_0 \cup N_2$. Since $\{d_n : n \in N_0 \cup N_2\}$ is dense in F_0 we obtain that $\psi_0[W_0] = F_0$.

Now we will find a clopen set W_1 with $\psi_0[W_1] = F_1$. However, we will have to be more careful because of possible intersections with W_0. Let $\mathcal{G}_1$ be a collection of clopen sets of ω^* such that $\bigcap \mathcal{G}_1 = \psi_0^{\leftarrow}[F_1]$ and $|\mathcal{G}_1| < \mathfrak{c}$. For each $n \in N_1$, let U_n be a clopen subset of ω^* such that $\psi_0[U_n] = \{d_n\}$. If $n \in N_2$, we choose two disjoint non-empty clopen subsets U_n^0 and U_n^1 of ω^* such that $U_n^0 \subset W_0$ and $\psi_0[U_n^i] = \{d_n\}$ for $i < 2$. Clearly, $U_n \subset \bigcap \mathcal{G}_1$ for $n \in N_1$ and $U_n^0 \cup U_n^1 \subset \bigcap \mathcal{G}_1$ for $n \in N_2$. So using **MA** and Lemma 2.1 again, we can find a clopen set $W_1 \subset \omega^*$ such that $W_1 \subset \bigcap \mathcal{G}_1$, $U_n \subset W_1$ for all $n \in N_1$, $U_n^1 \subset W_1$ for all $n \in N_2$ and $U_n^0 \cap W_1 = \emptyset$ for all $n \in N_2$. Again, it easily follows that $\psi_0[W_1] = F_1$.

So now consider $W_0 \setminus W_1$. If $n \in N_1$, $U_n \subset W_0 \setminus W_1$ so $d_n \in \psi_0[W_0 \setminus W_1]$. If $n \in N_2$, $U_n^0 \subset W_0 \setminus W_1$ so $d_n \in \psi_0[W_0 \setminus W_1]$. From this it follows that $\psi_0[W_0 \setminus W_1] = F_0$. Let $W = W_0 \cup W_1$, we now define $\psi_1 : W \to X_1$ such that

$$\psi_1(x) = \begin{cases} (\pi\restriction_{V_0})^{-1}(\psi_0(x)), & \text{if } x \in W_0 \setminus W_1, \text{ and} \\ (\pi\restriction_{V_1})^{-1}(\psi_0(x)), & \text{if } x \in W_1. \end{cases}$$

It is easy to see that ψ_1 is as required. □

5.2. Theorem Let X be a separable, compact, ED space. Then **MA** implies that there is a reversible filter $\mathcal{F}$ that is generated by a tower of height $\mathfrak{c}$ such that $K_{\mathcal{F}}$ is a P-set homeomorphic to X.

Proof. By our hypothesis, we may assume that $X \subset {}^{\mathfrak{c}}2$. For all pairs $\alpha \leq \beta \leq \mathfrak{c}$, let $\pi^{\beta}_{\alpha} : {}^{\beta}2 \to {}^{\alpha}2$ be the projection. Let $\{d_n : n < \omega\}$ be an enumeration of a countable dense set in X. By permuting the elements of $\mathfrak{c}$ and then renaming them if necessary, we may assume that if $n, m < \omega$ and $\pi^{\mathfrak{c}}_{\omega}(d_n) = \pi^{\mathfrak{c}}_{\omega}(d_m)$, then $n = m$. Let $X_\alpha = \pi^{\mathfrak{c}}_{\alpha}[X]$ for every $\alpha < \mathfrak{c}$.

We will recursively construct a decreasing sequence $\{K_\alpha : \omega \leq \alpha < \mathfrak{c}\}$ of clopen sets of ω^* and a sequence of continuous functions $\varphi_\alpha : K_\alpha \to X_\alpha$, for $\omega \leq \alpha \leq \mathfrak{c}$, in such a way that $\pi^{\beta}_{\alpha} \circ \varphi_\beta = \varphi_\alpha$ whenever $\omega \leq \alpha \leq \beta < \mathfrak{c}$. Once we have done this, let $K = \bigcap\{K_\alpha : \alpha < \mathfrak{c}\}$ and define $\varphi : K \to X$ by $\varphi(x) = \bigcup\{\varphi_\alpha(x) : \omega \leq \alpha < \mathfrak{c}\}$ for all $x \in K$. Notice that φ is continuous and $\pi^{\mathfrak{c}}_{\alpha} \circ \varphi = \varphi_\alpha$ for every $\omega \leq \alpha < \mathfrak{c}$.

Let $\{B_\alpha : \omega \leq \alpha < \mathfrak{c}\}$ be an enumeration of all clopen subsets of ω^*. Let $\{f_\alpha : \omega \leq \alpha < \mathfrak{c}\}$ be the collection of all bijections from ω onto itself such that each one is repeated cofinally often. Let Λ_0 be the set of infinite even ordinals $< \mathfrak{c}$ and let Λ_1 be the set of infinite odd ordinals $< \mathfrak{c}$. We will carry out our construction respecting the following conditions.

(a) $K_\omega = \omega^*$.
(b) For all $\omega \leq \alpha < \mathfrak{c}$ and $n < \omega$, $\varphi_\alpha[K_\alpha] = X_\alpha$.
(c) For all $\alpha \in \Lambda_0$, if $\varphi_\alpha[K_\alpha \cap B_\alpha] = X_\alpha$, then $K_{\alpha+1} \subset B_\alpha$.
(d) Let $\alpha \in \Lambda_1$. Assume that there exists a clopen sets $U \subset K_\alpha$ and $V \subset X_\alpha$ such that $\varphi_\alpha[U] = V$ and $\varphi_\alpha[f^*_\alpha[U]] \subset X \setminus V$. Then there is $x \in X_\alpha$ such that $f^*_\alpha[K_{\alpha+1} \cap \varphi^{\leftarrow}_{\alpha}(x)] \cap K_{\alpha+1} = \emptyset$.

It is not hard to prove that (c) implies that $\varphi : K \to X$ is irreducible, thus, a homeomorphism. And the proof that (d) implies that the filter $\mathcal{F}$ of neighborhoods of K is reversible is analogous to the corresponding one in the proof of Theorem 4.4. Since any separable subspace of ω^* is nowhere dense, we obtain that $\{K_\alpha : \alpha < \mathfrak{c}\}$ is a tower that generates $\mathcal{F}$. So we will only show how to carry out this construction.

Let $\beta < \mathfrak{c}$ be a limit ordinal, let us show how to find K_β and φ_β. Let $T = \bigcap\{K_\alpha : \alpha < \beta\}$ and define $\psi : T \to X_\beta$ by $\psi(x) = \bigcup\{\varphi_\alpha(x) : \omega \leq \alpha < \beta\}$ for all $x \in T$. Notice that ψ is continuous and **MA** implies that T is a regular closed set of ω^*. Because X_β has weight $\leq |\beta| < \mathfrak{c}$, $\psi^{\leftarrow}[\pi^{\mathfrak{c}}_{\beta}(d_n)]$ is an intersection of $< \mathfrak{c}$ many clopen sets for each $n < \omega$. By **MA**, we can choose for each $n < \omega$ a clopen set $U_n \subset T$ such that $\psi[U_n] = \{\pi^{\mathfrak{c}}_{\beta}(d_n)\}$. Then by considering the sets $\{U_n : n < \omega\} \cup \{K_\alpha : \alpha < \beta\}$, by **MA** and Lemma 2.1, we obtain that there is a clopen set $V \subset T$ such that $U_n \subset V$ for every $n < \omega$. Let $K_\alpha = V$ and $\varphi_\beta = \psi{\restriction}_V$.

Now assume that $\alpha < \mathfrak{c}$ and we want to define $K_{\alpha+1}$ and $\varphi_{\alpha+1}$. First, assume that $\alpha \in \Lambda_0$. Let $T = K_\alpha \cap B_\alpha$ if $\varphi_\alpha[K_\alpha \cap B_\alpha] = X_\alpha$; otherwise, let $T = K_\alpha$. Then $\varphi_\alpha[T] = X_\alpha$. Notice that $V_0 = \{x \in X_{\alpha+1} : x(\alpha) = i\}$ for $i < 2$ is a pair of clopen sets of $X_{\alpha+1}$ where $\pi^{\alpha+1}_{\alpha}$ is one-to-one and $X_{\alpha+1} = V_0 \cup V_1$. Thus, we can apply Lemma 5.1 to find a clopen set $W \subset T$ and a continuous function $\psi : W \to X_{\alpha+1}$ such that $\pi^{\alpha+1}_{\alpha} \circ \psi = \varphi_\alpha$. So let $K_\alpha = W$ and $\varphi_{\alpha+1} = \psi$.

Finally, assume that $\alpha \in \Lambda_1$. If the hypothesis of (d) does not hold, just use Lemma 5.1 like in the previous paragraph to define $K_{\alpha+1}$ and $\varphi_{\alpha+1}$. So assume otherwise. By **MA** and the fact that all points of X_α have character $\leq |\alpha| < \mathfrak{c}$, we may assume that for each $n < \omega$, there exists a clopen set $U_n \subset K_\alpha$ such that $\varphi_\alpha[U_n] = \{\pi^{c}_{\alpha}(d_n)\}$. Let N_0 be the set of $n < \omega$ such that $d_n \in V$. We may assume that $U_n \subset U$ for all $n \in N_0$. For $n \in \omega \setminus N_0$, we may assume that either $U_n \subset f^*_\alpha[U]$ or $U_n \cap f^*_\alpha[U] = \emptyset$. Let N_1 the set of all $n \in \omega \setminus N_0$ such that $U_n \subset f^*_\alpha[U]$ and $N_2 = \omega \setminus (N_0 \cup N_1)$.

For each $n \in N_1$, choose $p_n \in U_n$. Then $\{p_n : n \in N_1\}$ is a discrete (possibly empty) set contained in $f^*_\alpha[U]$. For each $n \in N_1$, let $q_n = (f^*_\alpha)^{\leftarrow}(p_n)$. Then $\{q_n : n \in N_1\}$ is a discrete set contained in U. Since no clopen subset of ω^* is separable, it is possible to choose, for each $n \in N_0$, $p_n \in U_n \setminus \mathrm{cl}_{\omega^*}(\{q_m : m \in N_1\})$. Then the set $\{p_n : n \in N_0\} \cup \{q_n : n \in N_1\}$ is a dicrete subset of U. Then, since countable subsets are C^*-embedded in ω^*, there exists a clopen set $W \subset U$ such that $\{p_n : n \in N_0\} \subset W$ and $\{q_n : n \in N_1\} \cap W = \emptyset$. With this, we can define

$$T = W \cup (K_\alpha \cap f^*_\alpha[U \setminus W]) \cup (\varphi_\alpha^{\leftarrow}[X \setminus V] \cap (K_\alpha \setminus f^*_\alpha[U]))$$

Clearly, T is a clopen subset of K_α. If $n \in N_0$, then $p_n \in W$ so $d_n \in \varphi_\alpha[T]$. If $n \in N_1$, then $p_n \in K_\alpha \cap f^*_\alpha[U \setminus W]$ so $d_n \in \varphi_\alpha[T]$. Finally, if $n \in N_2$, $U_n \subset \varphi_\alpha^{\leftarrow}[X \setminus V] \cap (K_\alpha \setminus f^*_\alpha[U])$ so $d_n \in \varphi_\alpha[T]$. Thus, $\{d_n : n < \omega\} \subset \varphi_\alpha[T]$, which implies that $\varphi_\alpha[T] = X$.

By an application of Lemma 5.1, there is a clopen set $T' \subset T$ and a continuous function $\psi : T' \to X_{\alpha+1}$ such that $\pi^{\alpha+1}_\alpha \circ \psi = \varphi_\alpha$. Let $K_{\alpha+1} = T'$ and $\varphi_{\alpha+1} = \psi$. Finally, choose $x \in V$ arbitrarily. Since $K_{\alpha+1} \cap \varphi_\alpha^{\leftarrow}(x) \subset T \cap \varphi_\alpha^{\leftarrow}(x) \subset W$, $f^*_\alpha[K_{\alpha+1} \cap \varphi_\alpha^{\leftarrow}(x)] \subset f^*_\alpha[W] \subset \omega^* \setminus T \subset \omega^* \setminus K_{\alpha+1}$. Thus, these choices satisty the conclusion of (d), so we have finished the proof. □

5.3. Question Is the conclusion of Theorem 5.2 still valid if X is not necessarily separable?

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