# REMARKS ON RECENT RESULTS IN SET-THEORETIC TOPOLOGY

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Abstract. We review several of our favorite results of the last several years.

## 1. Introduction

This paper has no theme beyond the desire to present very brief excerpts or outlines of a few of the most interesting of the heavily set-theoretic results in topology of the past few years. We are guided most by our own interests. We review some results about $N^*$ by Farah, about compact spaces of countable tightness by Rabus, Koszmider, and Eisworth. We introduce the reader to club guessing and present a new cardinal equality result of Arhangelskii and Buzykova.

## 2. Standard tools

Recall that $H(\theta)$ is the set of sets whose transitive closure has cardinality less than $\theta$. A statement of set-theory holds in a model $M$ if the statement with all quantifiers restricted to $M$ holds outright. The sets $H(\theta)$, constitute useful submodels of the universe of sets since each is a transitive model of most of the axioms of ZFC and given a space $X$ and property of interest of $X$, there is almost surely a sufficiently large $\theta$ so that for any statement of interest, if there is an (counter-)example then there is one in $H(\theta)$. In a similar way, set models, such as $H(\theta)$ have *elementary submodels*. These are subsets $M \prec H(\theta)$ with the property that any statement of set-theory using parameters from $M$ holds in $M$ exactly when it holds in $H(\theta)$.

If $P$ is a poset, then a $P$-name, $\tau \in V^P$, is recursively defined by $\tau \subset P \times V^P$. If $G$ is a generic filter, then $V[G] = \{val_G(\sigma) : \sigma \in V^P\}$ is the forcing extension by $P$ where $val_G(\sigma) = \{val_G(\tau) : (\exists p \in P)(p, \tau) \in \sigma\}$. A filter $G$ is said to be generic if $G \cap D \neq \emptyset$ for each dense set $D \subset P$, i.e. for each $p \in P$, there is a $d \in D$ such that $d \leq p$.

A sequence $\{a_\alpha : \alpha \in \omega_1\}$ is a $\diamondsuit$-sequence if for each $\alpha \in \omega_1$, $a_\alpha \subset \alpha$, and for all $X \subset \omega_1$, the set $S_X = \{\alpha : X \cap \alpha = a_\alpha\}$ is stationary. Another useful principle is $\clubsuit$ which is the statement obtained by demanding that $a_\alpha$ be cofinal in $\alpha$ for limit $\alpha$ and setting $S_X = \{\alpha : a_\alpha \subset X\}$.

The above is clearly not meant as a suitable introduction to the fundamental tools of this area of topology. There are by now many very adequate references for these basics, perhaps even including an article in [15].

A very interesting result is Shelah's club guessing [25]. This is a $\diamondsuit$-like principle on $\omega_2$ (and larger cardinals) which holds in ZFC. Let $S^2_0$ denote the set of ordinals in $\omega_2$ which have countable cofinality.

**Proposition 2.1.** *There is a sequence $\{c_\alpha : \alpha \in S^2_0\}$ where for each $\alpha \in S^2_0$, $c_\alpha$ is cofinal in $\alpha$ and for each cub $C \subset \omega_2$, there is a stationary set $S \subset S^2_0$ such that $c_\alpha \subset C$ for all $\alpha \in S$.*

and the proof is fun:

*Proof of Proposition 2.1.* Choose, for every $\alpha \in S^2_0$, an increasing and cofinal sequence $\langle s(\alpha, n) : n \in \omega\rangle$ in $\alpha$. If this results in a club-guessing sequence then we're done. If not then choose a club $D_0$ that is never guessed, put $C_0 = D'_0$ and define $s_0$ by $s_0(\alpha, n) = \sup D_0 \cap s(\alpha, n)$ if $\alpha \in C_0$ and $s_0(\alpha, n) = \sup D_0 \cap s(\alpha, n)$ otherwise.

We continue recursively, defining $C_\xi$ and $s_\xi$ for $\xi < \omega_1$, as follows. At any stage stop if $s_\xi$ defines a club-guessing sequence otherwise choose $D_{\xi+1}$ that is not guessed, set $C_{\xi+1} = D'_{\xi+1}$ and define $s_{\xi+1}(\alpha, n) = \sup D_{\xi+1} \cap s_\xi(\alpha, n)$ if $\alpha \in C_{\xi+1}$ and $s_{\xi+1}(\alpha, n) = s_\xi(\alpha, n)$ otherwise. At limit stages set $D_\xi = \bigcap_{\zeta<\xi} C_\zeta$ and $s_\xi(\alpha, n) = \min_{\zeta<\xi} s_\zeta(\alpha, n)$. Observe that $s_\xi(\alpha, n) \le s_\zeta(\alpha, n)$ whenever $\zeta < \xi$.

The construction stops before $\omega_1$ because otherwise consider the club set $C = \bigcap_\xi C_\xi$ and take $\alpha \in C$. Because $D_\xi$ was never guessed there will for every $\xi$ be an $n$ such that $s_{xi+1}(\alpha, n) < s_x i(\alpha, n)$. But then one $n$ will be chosen uncountably often, yielding an infinite decreasing sequence of ordinals. □

The notion of club guessing has been used in several important applications since its use in pcf theory by Shelah [26]. Examples of greatest interest to topologists (and most likely to illustrate its use) are the proof by Kojman and Shelah [18] that there is no universal linear order of cardinality $\omega_2$ if $\omega_2 < \mathfrak{c}$. Another nice application is the result of M. Džjamonja in [10] that if $\omega_2 < \mathfrak{c}$ then there is no universal uniform Eberlein compactum of weight $\omega_2$. This is a strengthening of one direction of Bell's result [6] that it is independent of ZFC that such spaces exists. Another very pretty application by Kojman and Shelah, [19], is in showing that $\mathfrak{h}(U(\aleph_\omega)) = \omega_1$ (see 3.1 for the definition of $\mathfrak{h}$).

Although the following result does not have independent interest, it does nicely illustrate the use of club-guessing which the reader will see is somewhat different than those from $\diamondsuit$.

Say that a madf $\mathcal{M}$ (of countable sets) on a cardinal $\kappa$ is embeddable into another, $\mathcal{L}$, if there is an injection of $\kappa$ so that each member of $\mathcal{M}$ is sent to a subset of a member of $\mathcal{L}$. Of course there are many possible concepts that one might reasonably call embeddability of a madf into another, but our purpose is to simply illustrate the concept of club-guessing.

**Proposition 2.2.** *There is a family $\{\mathcal{A}_\xi : \xi \in \mathfrak{c}\}$, of pairwise non-embeddable madf's on $\omega_2$.*

*Proof.* Fix the club-guessing sequence $\{c_\alpha : \alpha \in S^2_0\}$ as in Proposition 2.1, and let $c_\alpha = \{c^\alpha_n : n \in \omega\}$ be listed in increasing order.

The idea is to construct, for each infinite $X \subset \omega$ a madf $\mathcal{A}_X$ so that for each $\delta$ of countable cofinality

(1) for all $a \in \mathcal{A}_X$, $\{n : |a \cap [c^\delta_n, c^\delta_{n+1})| > 1\}$ is either finite or is equal to $X$, and
(2) there is an $a_\delta \in \mathcal{A}_X$ such that $a \cap [c^\delta_n, c^\delta_{n+1}) > 1$ for all $n \in X$.

It is pretty easy to see that $\mathcal{A}_X$ can be constructed. The application of club-guessing is in the proof that if $X \setminus Y$ is non-empty, then $\mathcal{A}_X$ cannot be embedded into $\mathcal{A}_Y$. Indeed, assume that $X \neq Y$ and assume that $f$ is an injection from $\omega_2$ into $\omega_2$. There is a cub $C$ on $\omega_2$ so that for each $\delta \in \omega_2$, $f^{-1}([0, \delta)) = \delta$. By the club-guessing principle, there is a $\delta \in S^2_0$ so that $c_\delta \subset C$. Note then that for any cofinal $a \subset \delta$, $f(a)$ is also cofinal in $\delta$. Let $a$ be a member of $\mathcal{A}_X$ with the property that $|a \cap [c^\delta_n, c^\delta_{n+1})| > 1$ exactly for each $n \in X$. Since $f$ is an injection and $c_\delta \subset C$, it follows that $|f(a) \cap [c^\delta_n, c^\delta_{n+1})| > 1$ for all $n \in X$. This, of course, implies that $f(a)$ cannot be contained in any $b \in \mathcal{A}_Y$. □

## 3. Linearly Lindelöf spaces

A.V. Arhangel′skii and R. Buzyakova proved a very nice result about linearly Lindelöf spaces, namely that first countable such spaces have cardinality at most $\mathfrak{c}$, [2]. Recall that a space $X$ is linearly Lindelöf if for each cover by a chain of open sets has a countable subcover – hence the term linear Lindelöf. This is known to be equivalent to the statement that for each regular uncountable $\kappa$ and each $\kappa$-sized subset of $X$ has a complete accumulation point (i.e. $(\kappa, \kappa)$-compact for each uncountable regular cardinal $\kappa$). Most such cardinal inequality results lend themselves naturally to a proof using elementary submodels, or as Arhangelskii calls it, the radial method. The core to the proof is Lemma 3.4 below, and it certainly does seem resistant to an elementary submodel approach. Theorem 3.2 may be a small improvement of the analogous result in [2] but otherwise all of these results are in [2]. We are curious about the possibility of removing the linearly Lindelöf assumption from Lemma 3.4.

**Definition 3.1.** For a space $X$, let $\mathrm{Ch}(X)$ be defined as the minimum $\kappa$ such that $\beta X \setminus X$ can be written as the union of at most $\kappa$ compact sets.

**Theorem 3.2.** *Suppose that $\kappa$ is a cardinal and $Y$ is a subspace of a space $X$ such that $\mathrm{Ch}(Y)$ and $|Y|$ are at most $\kappa$ and, for each $y \in Y$, $\chi(y, X) \le \kappa$, then $Y$ is a $G_\kappa$-set in $X$.*

*Proof.* Fix an elementary submodel $M$ of $H(\theta)$ with $|M| = \kappa$ and $Y, X$ are in $M$ with $\kappa \cup Y \subset M$. To see that $Y$ is a $G_\kappa$-set in $X$ we will show that $Y = \bigcap\{U \in M : Y \subset U \text{ and } U \subset X \text{ is open}\}$. Fix a family $\{K_\alpha : \alpha \in \kappa\}$ of compact subsets of $\beta Y \setminus Y$ so that $\beta Y \setminus Y = \bigcup\{K_\alpha : \alpha < \kappa\}$ as per the definition of $\mathrm{Ch}(Y)$. By elementarity, we can assume that $\{K_\alpha : \alpha \in \kappa\}$ is in $M$ because some such family is in $M$. Again elementarity can be applied to deduce that each $K_\alpha$ is a member of $M$ (since implicitly $\{K_\alpha : \alpha \in \kappa\}$ represents a function from $\kappa$ onto the family and with $\kappa \subset M$, the $\alpha$-th value of the function must also be in $M$. Suppose that $x \in X \setminus Y$ and let $\mathcal{F}_x = \{F \subset X : F \text{ is closed and } x \in F \in M\}$. We must show there is an $F \in \mathcal{F}_x$ such that $F \cap Y = \emptyset$. Assume otherwise and fix any $p \in \beta Y$ such that $p \in \overline{F \cap Y}$ for each member of the filter $\{F \cap Y : F \in \mathcal{F}_x\}$. Now, $p \notin Y$ since for each $y \in Y$, some neighborhood filter base for $y$ is contained in $M$ and thus there is an open $U$ in $M$ such that $y \in U$ and $x \in F = X \setminus U$. Therefore there is an $\alpha \in \kappa$ such that $p \in K_\alpha$. Since $K_\alpha$ is in $M$ and $|Y| \subset M$, there is an open cover of $Y$, $\mathcal{U}_\alpha$, which is in $M$ and a subset of $M$ such that for each $U \in \mathcal{U}_\alpha$, $K_\alpha \cap \overline{U}$ is empty. Of course, since $\mathcal{U}_\alpha$ is a cover of $Y$, $F = X \setminus \bigcup \mathcal{U}_\alpha$ is a closed set disjoint from $Y$. However by our assumption $F \notin \mathcal{F}_x$, hence there is a $U \in \mathcal{U}_\alpha$ such that $x \in U$, in which case $\overline{U} \in \mathcal{F}_x$. Of course this contradicts the assumption that $p \in \overline{\overline{U} \cap Y}$. □

**Lemma 3.3.** *If $X$ has countable tightness and is $(\omega_1, \omega_1)$-compact then for each countably complete filter of closed sets $\mathcal{F}$ and each $F \in \mathcal{F}$, there will be a countable set $D \subset F$ such that $\overline{D}$ meets each member of $\mathcal{F}$.*

*Proof.* Inductively choose points $x_\alpha \in F$ and elements $F_\alpha \in \mathcal{F}$ so that $F_\alpha \cap \overline{\{x_\beta : \beta \in \alpha\}}$ is empty. By always choosing $x_\alpha \in \bigcap\{F_\beta : \beta \leq \alpha\}$ we will be constructing a free sequence. Since $X$ has countable tightness, the sequence $\{x_\alpha : \alpha \in \omega_1\}$ can not have a complete accumulation point, contradicting that $X$ is $(\omega_1, \omega_1)$-compact. Therefore, for some $\alpha$, $D = \{x_\beta : \beta < \alpha\}$ will be as required. □

**Lemma 3.4.** *If $Y$ is first countable, separable and linearly Lindelöf, then* $\mathrm{Ch}(Y) \leq \mathfrak{c}$.

*Proof.* Let $p \in \beta Y$ and fix $\rho$ minimal such that there is a filter base $\{D_\alpha : \alpha < \rho\}$ of separable subsets of $Y$ such that $p \in F_\alpha = \overline{D_\alpha}$ for each $\alpha$ and $Y \cap \bigcap_{\alpha<\rho} F_\alpha$ is empty. We prove that $\rho$ is countable, which, together with $|Y| \leq \mathfrak{c}$, shows that $\beta Y \setminus Y$ is covered by $\mathfrak{c}$ such countable intersections. Since $Y$ is linearly Lindelöf, this $\rho$ must have countable cofinality. Let $\langle \rho_n : n \in \omega\rangle$ be an increasing cofinal sequence in $\rho$. Let $G_n = \bigcap\{F_\alpha : \alpha < \rho_n\}$. If some closed intersection of countably many neighborhoods of $p$ is disjoint from $G_n \cap Y$ then this family of neighborhoods together with $\{D_\alpha : \alpha < \rho_n\}$ would contradict the minimality of $\rho$. Therefore, by Lemma 3.3, there is a separable set $E_n \subset G_n \cap Y$ such that $p \in \overline{E_n}$. Now we obviously have that $\bigcap_{n\in\omega} \overline{E_n}$ is disjoint from $Y$ as required. □

We do not know if the assumption of linearly Lindelöf is required in the previous lemma. However, it can be used to give a simple proof of the more general result.

**Theorem 3.5.** *If $Y$ is a first countable linearly Lindelöf space of cardinality $\mathfrak{c}$, then* $\mathrm{Ch}(Y) \leq \mathfrak{c}$.

*Proof.* Fix an elementary submodel $M$ with $Y \cup M^\omega \subset M$ and $|M| = \mathfrak{c}$. We prove that $\beta Y \setminus Y$ is covered by the compact sets $K \subset \beta Y \setminus Y$ such that $K \in M$. Fix any $p \in \beta Y \setminus Y$. Let $\mathcal{F}_p$ denote the collection of zero-sets of $Y$ which are in $M$ and that have $p$ in their closure. If $\mathcal{F}_p$ is not countably complete then, since $M^\omega \subset M$, we have such a $K$ with $p \in K$ as the intersection of countably many closures of members of $\mathcal{F}_p$. If $\mathcal{F}_p$ is countably complete then by Lemma 3.3, there must be a countable $D \subset Y$ such that $p \in \overline{D}$. Of course $D \in M$ and by Lemma 3.4 there is a collection $\mathcal{K} \in M$ of compact sets covering $\overline{D} \cap (\beta Y \setminus Y)$ with $|\mathcal{K}| \leq \mathfrak{c}$. Since $\mathcal{K} \subset M$ there is some $K \in \mathcal{K} \cap M$ with $p \in K$ as required. □

**Theorem 3.6.** *If $X$ is first countable and linearly Lindelöf, then $X$ has cardinality at most $\mathfrak{c}$.*

*Proof.* Fix an increasing sequence $\{M_\alpha : \alpha \in \omega_1\}$ of elementary submodels, each of cardinality $\mathfrak{c}$, such that for each $\alpha$, $\{M_\beta : \beta < \alpha\} \cup [M_\alpha]^\omega$ are contained in $M_\alpha$. In addition, we assume that $X$ and its topology is a member of $M_0$, hence it suffices to prove that $M_{\omega_1}$ contains $X$. Let for $\alpha \in \omega_1$, $\mathcal{U}_\alpha$ equal the collection of all open $W \subset X$ such that $X_\alpha = X \cap M_\alpha$ is a subset of $W$. Note that $\mathcal{U}_\alpha$ is an element of $M_{\alpha+1}$ and by Lemma 3.2 and 3.5, $X_\alpha = \bigcap \mathcal{U}_\alpha$. Let $x \in X \setminus \bigcup_{\alpha<\omega_1} X_\alpha$ and $W_\alpha \in \mathcal{U}_\alpha$ so that $x \notin W_\alpha$. The sequence $\{W_\alpha : \alpha < \beta\}$ is a member of $M_\beta$ for each $\beta$ and so the open cover, $\{W_\alpha : \alpha < \omega_1\}$, of $\bigcup_{\alpha<\omega_1} X_\alpha$ has no countable subcover, contradicting that this closed set should be $\omega_1$-Lindelöf. □

## 4. Stone-Čech of N

Our first two main results in this section are from Farah's new book [13].

**Definition 4.1.** An ideal $J \subset P(N)$ is ccc over fin if there do not exist uncountably many almost disjoint members of $P(N) \setminus J$. Dually, we will also say that a closed set $K \subset N^*$ is ccc over fin, if for every disjoint open family of subsets of $N^*$, only countably many will meet $K$; equivalently, the ideal $J$, consisting of those $A \subset N$ such that $A^* \cap K = \emptyset$, is ccc over fin.

The next theorem is a special case for the ideal of finite sets of much more general results contained in [13]. In his terminology, the function $\varphi_h : P(N) \to P(N)$ defined by $\varphi(B) = h^{-1}(B)$ is an almost lifting of the homomorphism $F$. Both of the following two results are remarkable and have powerful applications. These results continue the investigation begun by Shelah [24] , Just [17], Shelah and Steprāns [28], and Velickovic [30]. Of course it started with Shelah's result that it is consistent that there are no non-trivial autohomeomorphisms of $N^*$.

**Theorem 4.2** (OCA and MA)**.** *If $F$ is a homomorphism from $P(N)/fin$ onto itself, then there is an $h$ from a subset $A$ of $\omega$ into $\omega$ so that $F(B) = h^{-1}(B)$ for all $B$ in some ccc over fin ideal $J$.*

**Theorem 4.3** (OCA and MA)**.** *If a subset of $N^*$ is a continuous image of $N^*$, then it is equal to the disjoint union of a clopen set and a nowhere dense set.*

This first sample application is proven in [9] and the second is an older result of W. Just [16] but it relates to a very interesting problem that is still open. One of the tricks to most applications of 4.2 is to show that there is a member of the ideal on which it is interesting to know that $F$ has a lifting. In an unrelated vein, many readers will also be interested to know that J.T. Moore has shown that that the conjunction with another version of OCA implies that $\mathfrak{c} = \omega_2$ [22].

**Theorem 4.4** (OCA and MA)**.** *$N^*$ does not map onto the Stone space of the measure algebra.*

**Theorem 4.5** (OCA and MA)**.** *The only $P$-sets of $N^*$ that are homeomorphic to $N^*$ are the clopen sets.*

*Proof.* Suppose that $K \subset N^*$ is a $P$-set and that $CO(K)$ is isomorphic to $P(N)/fin$. By theorem 4.3, we may assume that $K$ is nowhere dense (i.e. it has a relative clopen subset which is nowhere dense in $N^*$). Setting $F(A) = A^* \cap K$ for $A \subset \omega$, defines a homorphism onto $CO(K)$. Thus there is an ideal $\mathcal{I}$ which is ccc over fin such that $F$ is induced by $h^{-1}$ for each $B \in \mathcal{I}$. Now $K$ is homeomorphic to $N^*$, hence there is an uncountable family $\{B_\alpha : \alpha \in \omega_1\}$ of infinite subsets of $N$ such that $B_\alpha^* \cap K$ are pairwise disjoint. Since $K$ is a $P$-set and, for each $\beta < \alpha$, $(B_\alpha \cap B_\gamma)^*$ is disjoint from $K$, there is an infinite $C_\alpha \subset B_\alpha$ such that $C_\alpha \cap B_\beta$ is finite for each $\beta < \alpha$, and $C_\alpha^* \cap K \neq \emptyset$. Thus the $C_\alpha$'s are almost disjoint, hence there is some $\alpha$ such that $C_\alpha$ is in $\mathcal{I}$, i.e. $h$ induces the homomorphism. However this is impossible as can be seen as follows. Let $B = C_\alpha \cap h(A)$ where $A$ is given as in Theorem 4.2. Since $C_\alpha^* \cap K$ is not empty, it follows that $B$ is infinite. However as $K$ is nowhere dense, $B$ has an infinite subset $B_1$ such that $B_1^* \cap K$ is empty. It follows then that $F(B_1) = \emptyset \neq^* h^{-1}(B_1)$. □

If $D$ is any countable discrete subset of $N^*$, then $\overline{D}$ is homeomorphic to $\beta N$. Therefore, of course, there is a copy of $N^*$, namely $\overline{D} \setminus D$ which is ccc over fin but which is not itself ccc. In addition, there is a function $F$ from $P(N)/fin$ onto $P(D)/fin$ induced by $F(B) = \{d \in D : B \in d\}$. It is easily seen, just as in the case of the nowhere dense $P$-set, that there is no function $h : D \to N$ which induces $F$; the ideal $J$ of Theorem 4.2 would be $\{B \subset N : \overline{B} \cap D = \emptyset\}$

If a $P$-set is ccc over fin, then it is ccc, and as we see above, if $K$ is contained in a ccc set then $K$ is also ccc over fin. I do not know if the following is a theorem of ZFC. We use Kunen's notion of an $\mathfrak{c}$-OK point (see the article by Baker and Kunen in this volume) and such points are not the limit of any ccc subset.

**Proposition 4.6.** *It follows from $\diamondsuit$ that there is a closed set $K$ of $N^*$ which is ccc over fin but which is not contained in a ccc subset of $N^*$.*

*Proof.* Construct a sequence of $\mathfrak{c}$-OK points $\{x_\alpha : \alpha \in \omega_1\}$ in $N^*$ and infinite subsets, $b(\alpha, n)$, of $N$ as follows. Let $\{a_\beta : \beta \in \omega_1\}$ be an enumeration of $[\omega]^\omega$ and let $\{S_\alpha : \alpha \in \omega_1 \text{ and } \lim(\alpha)\}$ be a $\clubsuit$-sequence ($\diamondsuit$ is equivalent to $CH + \clubsuit$). For each limit $\beta$ and $\beta < \alpha < \omega_1$:

(1) there is a sequence $\langle \gamma_n : n \in \omega \rangle$ increasing cofinal in $\beta$ such that $b(\beta, n) \in x_{\gamma_n}$
(2) if there is $\langle \zeta_n : n \in \omega \rangle$ increasing cofinal in $\beta$ such that $S_\beta \supset \{\zeta_n : n \in \omega\}$ and $a_{\zeta_n} \in x_{\gamma_n}$, then $b(\beta, n) = a_{\zeta_n}$,
(3) $x_\alpha$ is in the closure of $\bigcup \{b(\beta, n)^* : n \in \omega\}$.

At stage $\alpha$, first check if $\alpha$ is a limit. If condition (ii) holds choose $b(\alpha, n)$ as required, if it does not let $b(\alpha, n) \in x_{\gamma_n}$ for any sequence $\gamma_n$ cofinal in $\alpha$. It is easy to check by induction that the family of cozero sets $\{C_\beta : \lim(\beta) \text{ and } \beta \leq \alpha\}$, where $C_\beta = \bigcup \{b(\beta, n)^* : n \in \omega\}$ has the finite intersection property. It is also easy to see that there are uncountably many $\omega_1$-OK points in $\bigcap \{\overline{C_\beta} : \lim(\beta) \text{ and } \beta \leq \alpha\}$ so we may choose $x_\alpha$ in this intersection distinct from $x_\beta$ for each $\beta < \alpha$.

To see that $K = \overline{\{x_\alpha : \alpha \in \omega_1\}}$ is ccc over fin assume that there is an uncountable $I \subset \omega_1$ such that the family $\{a_\gamma : \gamma \in I\}$ are almost disjoint and that for each $\gamma \in I$ there is a $\zeta_\gamma$ such that $a_\gamma \in x_{\zeta_\gamma}$. There is a cub $C \subset \omega_1$ such that for each $\delta \in C$ and each $\gamma \in \delta$, $\zeta_\gamma \in \delta$. In addition, we may assume that $I \cap \delta$ is cofinal in $\delta$ for each $\delta \in C$. Therefore there is a $\beta \in C$ such that $S_\beta \subset I$. By our construction, $\{b(\beta, n) : n \in \omega\}$ is a subset of $\{a_\gamma : \gamma \in S_\beta\}$ and for all $\zeta \in \omega_1 \setminus \beta$ and $a \in x_\zeta$, we have arranged that $a \cap b(\beta, n)$ is infinite for some $n$. Of course this implies that $I$ is countable. In addition, since each $x_\alpha$ is an $\mathfrak{c}$-OK point, no $x_\alpha$ is a limit point of a ccc subspace of $N^*$, hence $\{x_\alpha : \alpha \in \omega_1\}$ is not contained in a ccc subset of $N^*$. □

A copy of $\mathbb{N}^*$ in a space $X$ is said to be trivially occuring if there is a countable discrete set $D$ in $X$ such that $\mathbb{N}^* = \overline{D} \setminus D$, i.e., it is the result of embedding $\beta\mathbb{N}$ into $X$. Every copy of $\mathbb{N}^*$ in $I^\mathfrak{c}$ is trivially occuring, this follows from Tietze's extension theorem. What is not known is whether there is a copy of $\mathbb{N}^*$ in itself that is not trivially occuring. The following considerations lead to non-trivial copies of $\mathbb{N}^*$.

Consider the case that $X$ is the free union of a family $\{X_n : n \in \omega\}$ where each $X_n$ is a copy of the Cantor set (any compact ccc space with no isolated points would be interesting). A filter $\mathcal{F}$ on a family $\{X_n : n \in \omega\}$ is called *nice* if for each $F \in \mathcal{F}$, there are only finitely many $n$ such that $F \cap X_n$ is empty. Let us say that a nice filter is *maximal* if for each sequence $\{W_n : n \in \omega\}$ of clopen sets with $W_n \subset X_n$, there is an $F \in \mathcal{F}$, such that for each $n$, either $F \cap W_n$ is empty or $W_n \subset F$.

**Proposition 4.7.** *If there is a remote maximal nice filter on $X$, then there is a copy of $N^*$ which is not trivially occuring.*

It seems at least possible that MA implies there is a maximal remote nice filter on $X$.

## 5. Distributivity of $\mathbb{N}^* \times \mathbb{N}^*$

Another result we report on is the following result of Shelah and Spinas, see [27]. They prove that $\mathfrak{h}(N^* \times N^*) = \omega_1$ in the Mathias model (which is known to satisfy $\mathfrak{h}(N^*) = \omega_2$).

**Definition 5.1.** The cardinal number $\mathfrak{h}(X)$ is the minumim number of dense open sets whose intersection has empty interior. The number $\mathfrak{h}(\mathbb{N}^*)$ is usually denoted $\mathfrak{h}$ and is equal to the distributivity degree of $\mathcal{P}(\mathbb{N})/\mathit{fin}$.

In his MR review (MR2001f:03095) of the Shelah-Spinas paper, A. Blass writes:

> "This result follows fairly easily from two (difficult) propositions that are of considerable interest in their own right.

We were interested to see if the approach of [8] could also be used, that is, to determine a combinatorial property of a tree $\pi$-base which will guarantee that the forcing will not diagonalize any branches (new or old). We are not however, reproving or reproducing the other excellent results of [27]. The combinatorial property used in [8] for the reals was remote filter (a filter on a space $X$ is *remote* if for each nowhere dense set the filter has a member completely separated from it). Just as Blass says, the result follows easily once we know how to construct such a tree. We leave that aspect of the construction to the interested reader, we satisfy ourselves with re-proving that, if CH is assumed, then there is a tree $\pi$-base for $N^* \times N^*$ which is not diagonalized by the Mathias iteration.

**Theorem 5.2.** *In the model obtained by adding $\omega_2$ Mathias reals to a model of CH, $\mathfrak{h}(N^* \times N^*) = \omega_1 < \mathfrak{h} = \omega_2$.*

**Definition 5.3.** The Mathias poset $\mathbb{M}$ consists of pairs $(a, A)$ where $a \in [\omega]^{<\omega}$ and $A \in [\omega \setminus (\max a)]^\omega$. The ordering is defined by $(a, A) < (b, B)$ if $b \subset a$ and $a \cup A \subset b \cup B$.

The Mathias real, $\mathfrak{m}$, given by a generic filter $G \subset \mathbb{M}$, is equal to $\bigcup\{a : (\exists A)((a, A) \in G)\}$. The strictly increasing enumerating function $e_\mathfrak{m}$ of $\mathfrak{m}$, eventually dominates all the ground model functions $f \in \omega^\omega$. It is useful to notice that a condition $(a, A)$ precisely determines the values of $e_\mathfrak{m}(i)$ for each $i < |a|$ and leaves available exactly the values in $A$ for $e_\mathfrak{m}(|a|)$. In the iteration, let $\dot{g}_\alpha$ denote the enumerating function of the $\alpha$-th Mathias real.

The Mathias poset $\mathbb{M}$ has two important structural properties. The first is a consequence of Ramsey-like properties of $[\omega]^\omega$. It is called the *Prikry property* and it says that given $(a, A) \in M$ and some formula $\phi$ there is an infinite subset $B$ of $A$ such that $(a, B)$ decides $\phi$, i.e., either $(a, B) \Vdash \phi$ or $(a, B) \Vdash \neg\phi$. In practice this is often used in the following form: given a condition $(a, A)$, a name $\dot{x}$ and a finite set $F$ (from the ground model) then there is an infinite subset $B$ of $A$ such that either $(a, B) \Vdash (\dot{x} \notin F)$ or there is an $f \in F$ with $(a, B) \Vdash (\dot{x} = f)$.

Even though the counting function of the Mathias real is dominating we do have some control on the new reals: if $f \in {}^\omega\omega$ and if $\dot{x}$ is a name for a function such that

$p \Vdash (\forall n)(\dot{x}(n) < f(n))$ then there are a condition $q \leq p$ and a sequence $\langle F_n : n \in \omega \rangle$ of finite sets such that $p \Vdash (\dot{x}(n) \in F_n)$ and $|F_n| \leq 2^n$ for all $n$. This property is called the *Laver property* [24] and in it the function $n \mapsto 2^n$ can be replaced by any increasing unbounded function like $\log n$.

Now the next result is a consequence of the Laver property holding over the ground model $V[G_0]$ where $G_0 = G \cap P_1$ for a $P_{\omega_2}$-generic filter $G$ and $P_1$ would be isomorphic to $\mathbb{M}$. For convenience, let us treat the iteration as $\mathbb{M} * P_{\omega_2}$.

**Proposition 5.4.** *If $\{\dot{x}_n : n \in \omega\}$ is a $\mathbb{M} * P_{\omega_2}$-name of a subset of $\omega$ such that for each $n$, $1 \Vdash \dot{g}_0(n) < \dot{x}_n$, and $p \in \mathbb{M} * P_{\omega_2}$ is given, then there is an $q < p$ and a sequence $\{\dot{y}(n,i) : n \in \omega, i < 2^n\}$ of $\mathbb{M}$-names such that $q \Vdash \{\dot{x}_m : m \leq n\} \cap [g_0(n), g_0(n+1)) \subset \{\dot{y}(n,i) : i < 2^n\} \subset [g_0(n), g_0(n+1))$*

The next result is a standard approach using Mathias forcing and repeatedly applying the Prikry property so as to thin out the set $B$ until the desired $A$ is found after an induction of length $\omega$.

**Proposition 5.5.** *Let $\{\dot{y}(n,i) : n \in \omega, i < 2^n\}$ be a sequence of $\mathbb{M}$-names as in Proposition 5.4 and let $(b,B) \in \mathbb{M}$, then there is an $A = \{a_n : n \in \omega\} \subset B$ such that*

$$(\forall m \leq n)(\forall i < 2^m)(\forall k \leq n)(\forall e \subset a_n + 1) \text{ either}$$
$$(e, \{a_\ell : n < \ell \in \omega\}) \Vdash k = \dot{y}(m,i) \text{ or } (e, \{a_\ell : n < \ell \in \omega\}) \Vdash k \neq \dot{y}(m,i) \tag{5.1}$$

*and, for each $m \leq n$, $i < 2^m$ and $e \subset a_n + 1$ if there is $A' \subset A$ and $k$ such that $(e, A') \Vdash k = \dot{y}(m,i)$, then $k < a_{n+1}$.*

Once we have the name $\{\dot{x}_n : n \in \omega\}$ and the condition $(b,A)$ in the right form as given by the previous two results we can present the desired combinatorial property of the elements of the desired $\pi$-basis for the product $N^* \times N^*$.

**Proposition 5.6.** *Assume that $E, F$ are disjoint infinite subsets of $\omega$ such that for each $e \in E$ and $f \in F$, $|A \cap ((e,f) \cup (f,e))| > 2$ then $(b,A)$ forces that one of the following two sets, $\{n : E \cap \{\dot{y}(n,i) : i < 2^n\} \neq \emptyset\}$ or $\{n : F \cap \{\dot{y}(n,i) : i < 2^n\} \neq \emptyset\}$, is finite.*

*Proof.* Given any $(b_1, A_1) \leq (b, A)$ choose $A_2 \subset A_1$ so that either $a \in A_2$ implies $\min((E \cup F \cup A) \setminus a+1) \in E$ or $a \in A_2$ implies $\min((E \cup F \cup A) \setminus a+1) \in F$. Assume the former and we will show that $(b_1, A_2)$ forces that $F \cap \{\dot{y}(n,i) : i < 2^n\}$ is empty for all $n > \max(b_1)$. Fix any $(b_2, A_3) < (b_1, A_2)$ and assume for a contradiction that $(b_2, A_3) \Vdash \dot{y}(n,j) = f \in F$ for some $j \in 2^n$.

By the assumption on $(E,F)$, let $a_1 < a_2 < a_3 < f < a_4 < a_5 < a_6$ where $a_i \in A$ and $E \cap (a_1, a_6) = \emptyset$. Note then that $A_2 \cap [a_1, a_5)$ is empty. Since $(b_2, A_2) \Vdash \dot{y}(n,j) = f$, it follows that $|b_2 \cap f| \geq n$ (since $1 \Vdash \dot{g}_0(n) \leq \dot{y}(n,j)$ ) and, since $b_2 \cap f \subset a_1$, it follows that $n \leq a_1$.

By the construction of $A$ it follows that $(b_2 \cap a_4, A \setminus a_4+1)$ forces that $\dot{y}(n,j) = f$. Again since $b_2 \setminus b_1 \subset A_2$, and the properties of $A$, it follows that $(b_2 \cap a_2, A_2 \setminus a_2+1)$ forces that $\dot{y}(n,j) = f$ which is supposed to imply that $f < a_3$ - a contradiction. $\square$

One can complete the proof that $\mathfrak{h}(N^* \times N^*) = \omega_1$ by constructing (along with careful bookkeeping) a tree $\pi$-base for $N^* \times N^*$ so that for each name $\{\dot{y}(n,i) : n \in \omega, i < 2^n\}$ and $(b,A)$ as above, there is a maximal antichain of the tree consisting of pairs $(E,F)$ as in Proposition 5.6. Nonetheless, certainly Proposition 5.6 can be used to show that (assuming CH) there is a tree $\pi$-base for $N^* \times N^*$ which is not diagonalized by the iteration of Mathias forcing as follows. Fix an enumeration in order type $\omega_1$ of all the combinations of conditions $(b, A) \in \mathbb{M}$ and $\mathbb{M}$-names $\{\{\dot{y}(n,i) : i < 2^n\} : n \in \omega\}$ as in Proposition 5.5. The tree $\pi$-base would consist of pairs $(E, F)$ at level $\alpha$ which mod finite satisfy the condition of Proposition 5.6 with respect to all the names with index less than $\alpha$. Now consider any Mathias name $\{\dot{x}_n : n \in \omega\}$ of a subset of $N$ with the property that the pair $(\{\dot{x}_{2n} : n \in \omega\}, \{\dot{x}_{2n+1} : n \in \omega\})$ diagonalizes the tree. By passing to a subset we may assume that $g_0(n) < x_n$ for all $n$. Obtain the $\mathbb{M}$-names $\{\{\dot{y}(n,i) : i < 2^n\} : n \in \omega\}$ and $(b, A)$ from Propositions 5.4 and 5.5 for this name. Some extension $(b', A')$ of $(b, A)$ will have to force that $(\{\dot{x}_{2n} : n \in \omega\}, \{\dot{x}_{2n+1} : n \in \omega\})$ is mod finite below some $(E, F)$ from the tree on a level so as to satisfy 5.6 with respect to $(a, A)$ and $\{\{\dot{y}(n,i) : i < 2^n\} : n \in \omega\}$. But we have a contradiction from Proposition 5.6 since we will get that one of $E$ or $F$ is almost disjoint from $\{\dot{x}_n : n \in \omega\}$ since this latter set is forced to be contained in the union of $\{\{\dot{y}(n,i) : i < 2^n\} : n \in \omega\}$.

## 6. COUNTABLE TIGHTNESS IN COMPACT SPACES

The following two results were proven by the author and represent a strongly held interest in the topic of the section. It is a pleasure to report on brilliant improvements by M. Rabus [23], P. Koszmider [20] and T. Eisworth [11]. In the case of Eisworth (and Theorem 6.10), the result we include is building on an earlier paper by Eisworth and Roitman [12]. Each of these are major results and we can do little more than to outline the main ideas of the constructions.

This first result was also proven independently by van Douwen.

**Theorem 6.1** (CH). *Each initially $\omega_1$-compact space of countable tightness is compact.*

**Theorem 6.2.** *It is consistent with $MA(\omega_1)$ that every compact space of countable tightness contains a point of countable character.*

Rabus proved the following.

**Theorem 6.3.** *It is consistent that there is an initially $\omega_1$-compact space of countable tightness which is not compact.*

For each $L \subset \omega_2$ say that a Boolean algebra $B$ is $L$-minimal if there is $\{a_x : x \in L\} \subseteq B$ with the following properties:

(1) $B$ is generated by $\{a_x : x \in L\}$,
(2) if $x_1, \ldots, x_n < y$, then $a_y - \bigcup_{i \le n} a_{x_i} \neq \emptyset$,
(3) if $x < y$, then $a_x \cap a_y \in B_x$, where $B_x$ is the subalgebra of $B$ generated by $\{a_z : z \le x\}$.

The space we seek will be the Stone space of a particular $\omega_2$-minimal Boolean algebra. It is easy to describe the structure of the Stone spaces of such algebras. Let $B$ be a $\omega_2$-minimal algebra generated by $\{a_\alpha : \alpha \in \omega_2\}$ and let $X$ be its Stone space. The neighborhood base for the point $\alpha \in \omega_2$ is simply given by $\{a_\alpha - \bigcup_{\beta \in s} a_\beta : s \in [\alpha]^{<\omega}\}$, similarly, the neighborhood base for $\omega_2$ is obtained by treating $a_{\omega_2}$ as 1.

Suppose that $B$ is a $K$-minimal Boolean algebra, $K \in [\omega_2]^{<\omega}$, generated by $\{a_x : x \in K\}$. Of course $B$ is atomic with the atoms $\{b_0, b_1, \ldots, b_k\}$, where $b_{x_i} = a_{x_i} - \bigcup\{a_{x_j} : j < i\}$ for $i = 1, \ldots, k$ and $b_0$ is the complement of $\bigcup\{a_x : x \in K\}$.

When we speak of the atoms of $B$ we will actually be ignoring $b_0$ and be most interested in $\{b_1 \ldots, b_k\}$. We can look at an element $a_x$ as a union of some atoms of $B$. Since $B$ is minimal it follows that if an atom $b_y \in a_x$, then $y \leq x$ and we always have $b_x \in a_x$.

Let $L \subseteq K$. We say that $L$ generates a subalgebra of $B$ if $\{a_x : x \in L\}$ generates an $L$-minimal algebra. This is equivalent to say that for $x < y$ in $L$, the intersection $a_x \cap a_y$ is in the algebra generated by $\{a_v : v \leq x,\ v \in L\}$. Note that if $L$ is an initial part of $K$, then $L$ generates a subalgebra of $B$.

Suppose now that $K' = \{y_1, \ldots, y_k\}$ is another finite subset of the same size as $K$. Let $\Delta = K \cap K'$ and let $\{z_i : i \leq l\}$ be an enumeration of $\Delta$ in increasing order and let $B'$ be a $K'$-minimal algebra generated by $\{a'_y : y \in K'\}$ with the atoms $\{b'_1, \ldots, b'_k\}$. Suppose that the structures $(K, \Delta)$ and $(K', \Delta)$ are isomorphic, i.e, $\Delta$ has the same position in both $K$ and $K'$. Suppose also that the isomorphism between $(K, \Delta)$ and $(K', \Delta)$ extends to the isomorphism between $B$ and $B'$. Finally, assume that $\Delta$ generates a subalgebra of $B$, (thus also B').

Let $L = K \cup K'$, and define the minimal amalgamation of $B$ and $B'$ to be an $L$-minimal Boolean algebra $C$ generated by $\{c_x : x \in L\}$ such that $C$ restricted to $K$ is isomorphic to $B$ and $C$ restricted to $K'$ is isomorphic to $B'$. Moreover, the amalgamation is minimal in a sense that the intersection, $c_x \cap c_y$ where $x \in K - \Delta$, $y \in K' - \Delta$, is as small as possible. In particular if $\Delta = \emptyset$, then $c_x \cap c_y = \emptyset$.

For each $\alpha \in L$ let $d_\alpha$ be an atom. As noted above every $c_\alpha$ must be a union of some atoms $d_\beta$ with $\beta \leq \alpha$. We define some auxiliary sets. For $x \in K - \Delta$ let $D_x = \{d_x\}$, for $z \in \Delta$ let $D_z = \{d_y : y \in K', b'_y \in a'_z - \bigcup\{a'_v : v \in \Delta, v < z\}\}$ and $D'_z = \{d_x : x \in K, b_x \in a_z - \bigcup\{a_v : v \in \Delta, v < z\}\}$. For $y \in K' - \Delta$ let $D_y = \{d_y\}$.
We define $c_\alpha$ as follows. Suppose $\alpha \in K - \Delta$. If $v \in K$ and $b_v \in a_\alpha$, then put $D_v \subseteq c_\alpha$. In every other case put $D_v \cap c_\alpha = \emptyset$. Suppose $\alpha \in K' - \Delta$. Then $c_\alpha$ is a union of $D'_y$'s. If $w \in K'$ and $b'_w \in a'_\alpha$, then put $D'_w \subseteq c_\alpha$. In every other case $D'_w \cap c_\alpha = \emptyset$. Suppose now that $\alpha \in \Delta$. We define $c_\alpha$ as a union of some $D_z \cup D'_z$ for $z \in \Delta$, $z \leq \alpha$. If $z \leq \alpha$, $z \in \Delta$ and $b_z \in a_\alpha$ (so also $b'_z \in a'_\alpha$), then we put $D_z \cup D'_z \subseteq c_\alpha$. If $b_z \notin a_\alpha$, then put $(D_z \cup D'_z) \cap c_\alpha = \emptyset$.

A $\Delta$ function is a function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ with the following properties:

1. $f\{x, y\} \subseteq \min\{x, y\} + 1$ and $\min\{x, y\} \in f\{x, y\}$.
2. For all uncountable sets $D$ of finite subsets of $\omega_2$ there are $a, b \in D$, $a \neq b$ and $\forall x \in a - b\ \forall y \in b - a\ \forall z \in a \cap b$
   - (a) if $x, y > z$, then $z \in f\{x, y\}$,
   - (b) if $y > z$, then $f\{x, z\} \subseteq f\{x, y\}$,
   - (c) if $x > z$, then $f\{y, z\} \subseteq f\{x, y\}$.

It has been shown in [4], that a $\Delta$ function can be forced by a $\sigma$-closed $\omega_2$-cc poset $P$. In addition, Todorcevic has shown that such a function exists whenever there is a $\rho$-function as in [29, 5], hence the non-existence of a $\Delta$ function implies there are large cardinals.

We present now the ccc poset $Q$ which forces an $\omega_2$-minimal Boolean algebra $A$ generated by $\{a_\alpha : \alpha \in \omega_2\}$. A pair $(B, L)$ is a condition in $Q$ if $L = \{x_1, \ldots, x_k\}$ is a subset of $\omega_2$, and $B$ is a $L$-minimal Boolean algebra generated by some $\{c_x = B(x) : x \in L\}$. We will abuse notation and assume that $B$ denotes the algebra as well as the function which selects the generators. For the most part we will assume that the generators are clear from the context. The idea of using the $\Delta$ function is from [4] and is critical for the ccc property. For every $i, j \le k$ the element $c_{x_i} \cap c_{x_j}$ is in the Boolean algebra generated by $\{c_{x_m} : x_m \le \min\{x_i, x_j\}$ and $x_m \in f\{x_i, x_j\}\}$.

A condition $(B', L')$ extends $(B, L)$ if $L \subseteq L'$ and $L$ generates a subalgebra of $B'$ which is isomorphic to $B$.

**Lemma 6.4.** *The forcing $Q$ has the ccc.*

*Proof.* Let $\{(B_\alpha, L_\alpha) : \alpha \in \omega_1\}$ be an uncountable subset of $Q$. By thinning out we can assume that $\{L_\alpha : \alpha \in \omega_1\}$ form a $\Delta$-system with the root $\Delta$; for every $\alpha \ne \beta$, the structures $(L_\alpha, \Delta)$ and $(L_\beta, \Delta)$ are isomorphic and the isomorphism lifts to the isomorphism between $B_\alpha$ and $B_\beta$. Let $A \in [\omega_2]^\omega$ be such that $A$ is closed under $f$ and $\Delta \subseteq A$. Let $\alpha, \beta$, $\alpha \ne \beta$ be such that $L_\alpha \cap A = \Delta$ and $L_\beta \cap A = \Delta$ and $L_\alpha, L_\beta$ satisfy conditions $(a), (b), (c)$ of a $\Delta$-function. Note that it follows that $\Delta$ generates subalgebras of both $B_\alpha$ and $B_\beta$.

Let $L = L_\alpha \cup L_\beta$ and let $C$ be the minimal amalgamation of $B_\alpha$ and $B_\beta$. Suppose that $C$ is generated by $\{c_x : x \in L\}$. We have to prove that $(C, L)$ is a condition in $Q$. To see this it is enough to prove that if $x \in L_\alpha - \Delta$, $y \in L_\beta - \Delta$, $x < y$, then the intersection $c_x \cap c_y$ is in the algebra generated by $\{c_v : v \le \min\{x, y\}, v \in L \cap f\{x, y\}\}$. Assume e.g. that $x < y$. For $z \in \Delta$, $z < x$, the intersection $c_x \cap c_z$ is in the algebra generated by $\{c_v : v \le z, v \in L_\alpha \cap f\{z, x\}\}$, and also $c_y \cap c_z$ is in the algebra generated by $\{c_v : v \le z, v \in L_\beta \cap f\{z, y\}\}$. Note that we have that $f\{x, z\} \cup f\{y, z\} \subseteq f\{x, y\}$. Therefore the above intersections are in the algebra generated by $\{c_v : v \le x, v \in L \cap f\{x, y\}\}$. Similarly we show that for $w \in \Delta$, $x < w < y$ we have $c_x \cap (c_w - \bigcup\{c_v : v < w, v \in \Delta\})$ is in the algebra generated by $\{c_v : v \le x, v \in L \cap f\{x, y\}\}$. This follows from the fact that $f\{x, v\} \subseteq f\{x, y\}$ for $v \in \Delta$, $v < y$. □

Note that $Q$ forces an $\omega_2$-minimal algebra $A$. If $G$ is $Q$-generic, then for each $\alpha \in \omega_2$, $a_\alpha = \{\beta \le \alpha : (\exists (B, L) \in G)\ (\beta, \alpha \in L$ and $B(\beta) \in B(\alpha))\}$. The desired space is $\omega_2$ with the topology induced by the Stone space of $B$, i.e. $S(B) \setminus \omega_2$. It is reasonably straightforward to show that $S(B)$ has countable tightness and that $\omega_2$ is not the unique limit of any $\omega_1$-sequence in $\omega_2$ (see [23]). These properties imply that $S(B) \setminus \{\omega_2\}$ is then initially $\omega_1$-compact. The following result is the most important new idea which distinguishes it from the algebra in [4]. As we will see, the proof (lifted liberally from [23]) is very involved. The idea is roughly as follows. A condition $q$ is chosen that decides a clopen neighborhood of a sufficiently large $\gamma$ which is disjoint from a countable sequence. The condition $q$ is *reflected* to an isomorphic condition below $\gamma$ which in turn is extended to absorb a member of the sequence. The key step is the invention of a way to extend this new condition in such a way to guarantee that $q$ is compatible with an extension that allows $a_\gamma$ to be maximal (rather than the usual minimal).

**Lemma 6.5.** *There are no $\omega$-sequences in $X$ converging to $\omega_2$.*

Its proof depends on the following property of the $\Delta$-function $f$.

**Lemma 6.6.** *Let $\gamma \in \omega_2$, $cf(\gamma) = \omega_1$. Let $B \in [\gamma]^\omega$, $E \in [\omega_2 - \gamma]^{<\omega}$, $E = \{x_1, \ldots, x_k\}$. Then there is $A \in [\gamma]^\omega$ containing $B$ and $E' \in [\gamma - \delta]^k$, $E' = \{x'_1, \ldots, x'_k\}$ and $F \in [[\gamma]^2]^k$ such that $F = \{\{y^1_i, y^2_i\} : i \le k\}$ and letting $D = E' \cup \bigcup F$ we have*

*(1) For all $i, j \le k$, $\sup(A) < x'_i < y^1_j < y^2_j$.*

(2) *For* $i < j$, $y_i^2 < y_j^1$.
(3) *If* $z \in D \cup A$ *and* $y \in A$, *then* $f\{z, y\} \subseteq A$.
(4) *For* $i \leq k$, *if* $z$ *is any element of* $D$ *with a subscript* $i$, *then* $f\{x_i, y\} = f\{z, y\}$ *for every* $y \in A$.
(5) *For* $i \leq k$, *for every* $v \in D$, $(A \cup \{z : z \in D, z < v\}) \subseteq f\{x_i, v\}$.
(6) *For every* $v, w \in D$, *if* $v < w$, *then* $(A \cup \{z : z \in D, z < v\}) \subseteq f\{v, w\}$.

We also recall the following from [23].

**Lemma 6.7.** *Let* $A \subseteq \omega_2$ *be closed under* $f$. *Suppose that* $q_1 = (B_1, L_1)$ *and* $q_2 = (B_2, L_2)$ *are two conditions in* $Q$ *such that* $L_1 \subseteq A$, $D = L_2 - A$ *is such that if* $y \in A$ *and* $z \in D$ *then* $y < z$ *and* $A \cup D$ *is closed under* $f$. *Suppose that a condition* $q = (B, L)$ *extends both* $q_1$ *and* $q_2$. *Then the restriction of* $q$ *to* $A \cup D$ *is a condition in* $Q$ *extending* $q_1$ *and* $q_2$.

*Proof.* of Lemma 6.5. Suppose that $\{x_n : n \in \omega\} \subseteq \omega_2$ is a sequence converging to $\omega_2$. We work in the model $V^P$. For $n \in \omega$ let $A_n$ be a maximal antichain in $Q$ that determines $x_n$, i.e., for each $u \in A_n$ there is some $\alpha_n \in L(u)$ such that $u \Vdash x_n = \alpha_n$. Let $\gamma \in \omega_2$, $\mathrm{cf}(\gamma) = \omega_1$ be such that for every $n$, for every $u \in A_n$, $L(u) \subseteq \gamma$.

Since $a_\gamma$ is a clopen set avoiding the point $\omega_2$, there is some condition $q \in Q$ and $m \in \omega$ such that $q \Vdash x_n \notin a_\gamma$ for every $n > m$. Let $q = (B(q), L(q))$ and let $B(q)$ be generated by $\{a_x(q) : x \in L(q)\}$ with the atoms $\{d_x(q) : x \in L(q)\}$. We can assume that $\gamma \in L(q)$. Then $L(q) = L' \cup E$, where $L' = L(q) \cap \gamma$, $E = L(q) - L'$. Let $\{x_1, \ldots, x_k\}$ be an enumeration of $E$ in the increasing order. Note that $x_1 = \gamma$. Let $A \in [\gamma]^\omega$ be such that $A$ contains $L(q) \cap \gamma$ and $supp(u)$ for every $u \in A_n$, $n \in \omega$ and let $E', F$ be as in Lemma 6.6.

Let us now give the idea of the rest of the proof. Our intention is, of course, to find a condition $r \leq q$ and $n > m$ such that $r \Vdash x_n \in a_\gamma$. To do this we first find a condition $s$, such that $L(s) = L' \cup E'$, and $B(s)$ is isomorphic to $B(q)$ via bijection from $E$ to $E'$.

Let $U$ be a name for $\bigcup\{a_x : x \in L(s)\}$. If $p \in Q$ is any condition such that $L(s) \subseteq L(p)$, then by $U(p)$ we denote $\bigcup\{a_x(p) : x \in L(s)\}$. In particular $p \Vdash U(p) = U$. Next we find an auxiliary condition $t \leq s$ such that $L(t) = L(s) \cup \bigcup F$, $B(t)$ restricted to $L(s)$ is isomorphic to $B(s)$. Since $t$ forces that $U$ is a clopen set disjoint from $\{\omega_2\}$ and $\{x_n : n \in \omega\}$ converges to the point $\omega_2$, we can find a condition $t' \leq t$ and $n > m$ such that $t' \Vdash x_n \notin U$. We can assume that $t'$ extends some condition $u \in A_n$, so there is some $\alpha_n \in L(t')$ such that $t' \Vdash \alpha_n = x_n$. By Lemma 6.7, since $L(u) \subseteq A$, we can assume that $L(t') - L(t) \subseteq A$. Let $W$ be a name for $\bigcup\{a_x(t') : x \in L(t')\}$. For any condition $p$ such that $L(t') \subseteq L(p)$, let $W(p) = \bigcup\{a_x(p) : x \in L(t')\}$.

Finally we define $r \leq t'$. Our intention is to define $r$ such that $r$ extends $q$ and $W(r) - U(r) \subseteq a_\gamma$. Then $r \Vdash x_n \in a_\gamma$, a contradiction.

To finish the proof we show that $s$ and $t$ are conditions in $Q$, then we define $r$ and show that it has required properties.

Recall that $L(s) = L' \cup E'$ and the bijection between $L(s)$ and $L(q)$, constant on $L'$, lifts to an isomorphism of $B(s)$ with $B(q)$. Hence for $y < x$ in $L(s)$, if $y, x \in L'$, then $d_y(s) \in a_x(s)$ iff $d_y(q) \in a_x(q)$. If $y \in L'$, $x \in E'$, then $x = x'_i$ for some $i \leq k$ and $d_y(s) \in a_{x'_i}(s)$ iff $d_y(q) \in a_{x_i}(q)$. Finally, if $y, x \in E'$, then $y = x'_j$, $x = x'_i$ and $d_{x'_j}(s) \in a_{x'_i}(s)$ iff $d_{x_j}(q) \in a_{x_i}(q)$.

We check that $s$ is a condition in $Q$. Let $y < x$ in $L(s)$. We have to show that $a_x(s) \cap a_y(s)$ is in the algebra generated by $\{a_v(s) : v \leq y,\ v \in L(s) \cap f\{x, y\}\}$. If $x, y \in L'$ we have nothing to do since the isomorphism is constant on $L'$. Assume that $y \in L'$, $x = x'_i$. Then $a_y(s) \cap a_{x'_i}(s)$ has the same representation by $\{a_v(s) : v \leq y\}$ as $a_y(q) \cap a_{x_i}(q)$ by $\{a_v(q) : v \leq y\}$. Moreover, since $q$ is a condition the intersection $a_y(q) \cap a_{x_i}(q)$ is in the algebra generated by $\{a_v(q) : v \in L(q) \cap f\{x_i, y\}\}$. By Lemma 6.6(4), we have $f\{x_i, y\} = f\{x'_i, y\}$, hence the intersection $a_y(s) \cap a_{x'_i}(s)$ is in the algebra generated by $\{a_v(s) : v \in L(s) \cap f\{x'_i, y\}\}$.

If $y = x'_j$, $x = x'_i$ with $j < i$, then by Lemma 6.6(6), $\{v : v \in L(s),\ v \leq y\} \subseteq f\{x'_i, x'_j\}$, hence $a_{x'_j}(s) \cap a_{x'_i}(s)$ is in the algebra generated by $\{a_v(s) : v \in L(s) \cap f\{x'_i, x'_j\}\}$ and we are done.

Now we define $t$, extending $s$. Let $L(t) = L(s) \cup \bigcup F$. We define $B(t)$ such that $B(t)$ restricted to $L(s)$ is isomorphic to $B(s)$. We define $a_{y_i^1}(t)$ and $a_{y_i^2}(t)$ for $i = 1, \ldots, k$ such that $a_{y_i^\mu}(t) \cap a_{y_j^\nu}(t) = U(t)$ for every $(\mu, i) \neq (\nu, j)$. Recall that $U(t) = \bigcup_{v \in L(s)} a_v(t)$. To check that $t$ is a condition note that if $(\mu, i) \neq (\nu, j)$, then by Lemma 6.6(6), $L(s) \subseteq f\{y_i^\mu, y_j^\nu\}$.

Recall that $t' \leq t$ is a condition such that $L(t') - L(t) \subseteq A \cap \gamma$. Finally we define $r$. Put $L(r) = L(t') \cup E$. The algebra $B(r)$ restricted to $L(t')$ is isomorphic to $B(t')$. We have to define $a_{x_i}(r)$ for $i = 1, \ldots, k$. Recall that $W(r) = \bigcup\{a_x(r) : x \in L(t')\}$. For $x \in L(q)$ we define auxiliary sets $D_x$ as follows. If $x \in L'$, then put $D_x = a_x(r) - \bigcup\{a_y(r) : y < x,\ y \in L'\}$. Assume $x \in E$, i.e., $x = x_i$ for some $i$. Assume first that $i = 1$. Put $D_{x_1} = (a_{x'_1}(r) - \bigcup\{a_y(r) : y < x'_1,\ y \in L(s)\}) \cup (W(r) - U(r)) \cup \{d_{x_1}(r)\}$. For $i > 1$ define $D_{x_i}(r) = (a_{x'_i}(r) - \bigcup\{a_y(r) : y < x'_i,\ y \in L(s)\}) \cup \{d_{x_i}(r)\}$.

Now define $a_{x_i}(r)$ for $i \leq k$. Each $a_{x_i}(r)$ is a union of some sets $D_x$, $x \in L(q)$. We put $D_x \subseteq a_{x_i}(r)$ if $d_x(q) \in a_{x_i}(q)$, otherwise we put $D_x \cap a_{x_i}(r) = \emptyset$. Note that this is well defined since the sets $D_x$ are pairwise disjoint.

*Claim* 1. For $x_i \in E$, $a_{x_i}(r) \cap U(r) = a_{x'_i}(r)$. Moreover $a_{x_i}(r) \cap W(r) = a_{x_i}(r) \cap U(r)$ if $d_{x_1}(q) \notin a_{x_i}(q)$ and $a_{x_i}(r) \cap W(r) = (a_{x_i}(r) \cap U(r)) \cup (W(r) - U(r))$ if $d_{x_1}(q) \in a_{x_i}(q)$.

*Proof.* Note that $a_{x_i}(r) \cap U(r)$ is a union of sets of the form $D'_x = a_x(r) - \bigcup\{a_y(r) : y < x,\ y \in L(s)\}$, for $x \in L(s)$. Since $B(r)$ restricted to $L(s)$ is isomorphic to $B(s)$ and the bijection between $E'$ and $E$ lifts to the isomorphism between $B(s)$ and $B(q)$, it follows that $a_{x'_i}(r)$ is also a union of sets $D'_x$ for $x \in L(s)$. Moreover $D'_x \subseteq a_{x'_i}(r)$ if and only if $D'_x \subseteq a_{x_i}(r) \cap U(r)$. The second part is obvious by the definition of $a_{x_i}(r)$.

Consider now $B(r)$ restricted to $L(q)$. For $x \in L(q)$ the set $a_x(r) - \bigcup\{a_y(r) : y < x,\ y \in L(q)\}$ is equal to $D_x$. By the definition, $D_y \subseteq a_x(r)$ if $d_y(q) \in a_x(q)$, and $D_y \cap a_x(r) = \emptyset$ otherwise. Hence the restriction of $B(r)$ to $L(q)$ is isomorphic to $B(q)$.

Finally we have to show that $r$ is a condition in $Q$, i.e., we have to show that if $y < x$ in $L(r)$, then $a_x(r) \cap a_y(r)$ is in the algebra generated by $\{a_v(r) : v \in L(s) \cap f\{x, y\}\}$. Since $B(r)$ restricted to $L(t')$ is isomorphic to $B(t')$ we can assume that $x \in E$, i.e., $x = x_i$ for some $i \leq k$.

Assume first that $y \in L(t') - (E' \cup \bigcup F)$. By the definition, since $a_y(r) \subseteq W(r)$ for $r \in L(t')$ we have $a_{x_i}(r) \cap a_y(r)$ is equal to either $a_{x'_i}(r) \cap a_y(r)$ or $(a_{x'_i}(r) \cap a_y(r)) \cup (a_y(r) - U(r))$. Recall that $U(r) = a_{y_i^1}(r) \cap a_{y_i^2}(r)$. Hence $a_y(r) - U(r) =$

$a_y(r)-((a_y(r)\cap a_{y_i^1}(r))\cap(a_y(r)\cap a_{y_i^2}(r)))$. Moreover, since $(L(t')-(E'\cup\bigcup F))\subseteq A$, it follows by Lemma 6.6(4) that the sets $f\{y,x_i\}$, $f\{y,x_i'\}$, $f\{y,y_i^1\}$, $f\{y,y_i^2\}$ are equal and we are done.

Assume that $y\in E'$. Then $y=x_j'$ for some $j\le k$ and then, since $a_{x_j'}(r)\subseteq U(r)$, we have $a_{x_i}(r)\cap a_{x_j'}(r)=a_{x_i'}(r)\cap a_{x_j'}(r)$. But, by Lemma 6.6 (5), we have $\{v: v\in L(t'), v\le x_j'\}\subseteq f\{x_i,x_j'\}$ and of course $a_{x_i'}(r)\cap a_{x_j'}(r)$ is in the algebra generated by $\{a_v(r): v\in L(t'), v\le x_j'\}$.

Finally, if $y\in\bigcup F$, say $y=y_j^1$, then $a_{x_i}(r)\cap a_{y_j^1}(r)$ is equal either to $a_{x_i'}(r)$ or $a_{x_i'}(r)\cup(a_{y_j^1}(r)-U(r))$. In the first case note that $x_i'\in f\{x_i,y_j^1\}$. In the second case recall that $U(r)=\bigcup\{a_v(r): v\in L(s)\}$ and $L(s)=L'\cup E'$. Hence $a_{y_j^1}(r)-U(r)=a_{y_j^1}(r)-\bigcup\{a_{y_j^1}(r)\cap a_v(r): v\in L(s)\}$. By Lemma 6.6 (5) it follows that $L(s)\subseteq f\{y_j^1,x_i\}$. Hence $r$ is a condition in $Q$. This finishes the proof of the lemma. □

This next result is again an impressive forcing construction by Koszmider [20]. Spaces of countable tightness in which there are no points of first countability have been constructed from $\diamondsuit$ (Fedorcuk [14]) and consistent with any cardinal arithmetic (Malyhin [21]). However this is the first example, in any model, of a first countable space with such a continuous image.

**Theorem 6.8.** *It is consistent (with MA) that there is a first countable space which maps onto a space (of countable tightness) in which there are no points of countable character.*

The paper [20] explores many applications of the following forcing

$$P(A,u)=\{(p_{-1},p_1)\in[A]^2:(p_{-1}\cup p_1)\notin u, p_{-1}\cap p_1=\emptyset\}$$

where $A$ is a Boolean algebra and $u$ is an ultrafilter on $A$. The effect of the forcing is to add a generic element $g$ so that in $A\cup\{g\}$ the only ultrafilter which is split is $u$. There is complete symmetry, but for definiteness, we take $g=\bigcup\{p_1:(\exists p_{-1})(p_{-1},p_1)\in G\}$.

These forcings are not, in general, ccc; in fact, if they are, the Stone space has countable tightness. If $P(A,u)$ is ccc, then

$$\forall a\notin u\ |A\restriction a|\le\omega$$

and $t(A)=\omega$. Moreover if $A$ is a Boolean algebra such that for each ultrafilter $u$ of $A$ there is a subalgebra $A_u\subset A$ such that $u$ is generated from $A_u$ and $P(A_u,A_u\cap u)$ is ccc, then $t(A)$ is countable. The plan is to construct such an $A$ so that the character of each ultrafilter $u$ is uncountable.

Slightly more is required to get the first countable preimage but this is the most interesting part. The construction actually ensures that the countable finite support product, $P^\omega(A_u,A_u\cap u)$, is ccc and of cardinality $\omega_1$. If, for each $n$, $a_n$ is the generic element added by the $n$-th coordinate of the product then consider the algebra generated by $A'=A\cup\{a_n:n\in\omega\}$. Every ultrafilter of $A'$ which extends the filter $u$ has countable character and every other ultrafilter $v$ of $A$ remains an ultrafilter in $A'$. The genericity over the product $P^\omega(A_u,A_u\cap u)$ guarantees that for distinct ultrafilters $v_1,v_2\ne u$ on $A_u$, there will be an $n$ such that $a_n\in v_1$ and $a_n\notin v_2$ and that $\{a_n:n\in\omega\}$ forms an independent family. That is, the closed set $K$ in the Stone space of $A'$ consisting of the ultrafilters extending $u$ is a copy of the Cantor set, and for each such $k \in K$, there's at most one $v \in S(A_u) \setminus \{u\}$ such that for all $n$, $a_n \in k \cap v$ or $-a_n \in k \cap v$. Using Martin's Axiom, we so extend $A$ for each ultrafilter $u$.

A careful structure is needed for the algebra $A$ which will use complete branching subtrees of $\{-1,1\}^{<\omega_1}$, $T$. A $T$-algebra is a Boolean algebra with a set of generators $\{a_t : t \in T\}$ such that for each $t \in T$

(1) $\{a_s : s < t\}$ is a filter and $a_t$ is minimal for $(A_t, u_t)$ where $A_t$ is the subalgebra generated by $\{a_s : s < t\}$ and $u_t$ is the filter generated by $\{a_s : s < t\}$.
(2) $a_{t^\frown -1} = -a_{t^\frown 1}$.

**Lemma 6.9.** *If $u$ is any ultrafilter on $A$, there is a maximal branch $b$ of $T$ such that $u = u_b$ is generated by $\{a_t : t \in b\}$. Conversely, for each such maximal branch $b$, $u_b$ is an ultrafilter.*

Notice that the minimum character is at least the cofinality of a maximal branch. By induction on $\alpha < \omega_2$, define

(1) a finite support iteration of forcings $P_\alpha$ of length $\alpha$ such that $P_\alpha \restriction \beta = P_\beta$ for each $\beta < \alpha$.
(2) A $P_\alpha$-name for a tree $\dot{T}_\alpha$ such that $\dot{T}_\beta \prec \dot{T}_\alpha$ for each $\beta < \alpha$.
(3) A $P_\alpha$-name $\dot{A}_\alpha = \langle \dot{a}_t : t \in \dot{T}_\alpha \rangle$ for a $\dot{T}_\alpha$-algebra such that $\dot{A}_\beta \prec \dot{A}_\alpha$ for each $\beta < \alpha$.

$P_{\beta+1} = P_\beta * (\dot{P}^\beta * \dot{Q}^\beta)$ where $\dot{P}^\beta$ is $\Pi\{P^\omega(A_{\dot{b}}, u_{\dot{b}}) : \dot{b}$ is a maximal branch of $\dot{T}_\beta\}$ and, of course $P^\omega(A,u)$ is simply the finite support product of countably many copies of $P(A,u)$.

If we want MA to hold we must let $\dot{Q}^\beta$ be a suitably chosen (Souslin free (see [7])) ccc poset. For our purposes we are satisfied to skip this part; the Souslin free is necessary to maintain control over when new uncountable branches are added to the $\dot{T}_\alpha$. Note though, that we are including $\dot{b}$ which are uncountable maximal branches of $\dot{T}_\beta$ and so we will have the necessary generics for the first countable extension.

We refer the reader to [20] for the proof that $P_{\omega_2}$ is ccc although we do outline some of the main ideas. The most problematic is to analyze the product of those $P(A_b, u_b)$ where $b$ is an uncountable branch. Again, we are only giving a very superficial overview which will give the flavor.

If $\mathrm{cf}(\alpha) = \omega_1$, then for each $\dot{b}$, a new uncountable branch of $\dot{T}_\alpha$, there is a function $f_b : \omega_1 \to \alpha$ (cofinal) such that for each $\xi \in \omega_1$, we have

$$b \restriction \xi \in V^{P_{f(\xi)}} \wedge b \restriction \xi \text{ is maximal}$$

(or of course $\dot{b}$ might be a subset of $T_\beta$ but the forcing ensures that such $\dot{b}$ first occur in $V^{P_\beta}$).

If the branches in the proof of ccc come from a fixed $V^{P_\beta}$, then the corresponding $P(A,u)$ have become $\sigma$-centered hence cause no difficulty. So we assume that we have those functions $f$ and we work in the forcing extension by $P_\alpha$.

Fix a maximal antichain $X$ included in the product. There is a non-decreasing function $f$ from $\omega_1$ into $\beta$ so that for each $\xi \in \omega_1$, $X \restriction \xi$, defined as

$$X \cap [P^n(A_{b_1 \restriction \xi}, u_{b_1 \restriction \xi}) \times \ldots \times P^n(A_{b_k \restriction \xi}, u_{b_k \restriction \xi})],$$

we have $X \restriction \xi \in V^{P_{f(\xi)}}$ and for each $\xi$ in a cub $C$,

$$X \restriction \xi \text{ is a maximal antichain in } P^n(A_{b_1 \restriction \xi}, u_{b_1 \restriction \xi}) \times \cdots \times P^n(A_{b_k \restriction \xi}, u_{b_k \restriction \xi}) \ .$$

Of course we can find a $\xi \in C$ such that $f_{b_i}(\xi) = f(\xi) = \theta$ for each $i$ and so that all the branches $b_i$ have split. Then by induction on $\eta$ for $\theta \leq \eta < \beta$ one can prove that $X \restriction \xi$ is a maximal antichain in $P^n(B^\eta_1, \langle u_{b_1 \restriction \xi}\rangle^{Fi}) \times \cdots \times P^n(B^\eta_k, \langle u_{b_k \restriction \xi}\rangle^{Fi})$ where $B^\eta_i = \langle \{a_t : t \in T_\eta, b_i \restriction \xi \subset t\}\rangle$ and $\langle u\rangle^{Fi}$ denotes the filter generated by $u$.

Once this has been shown, closer examination shows that

$$[P^n(A_{b_1}, u_{b_1}) \times \ldots \times P^n(A_{b_k}, u_{b_k})] \,,$$

is a subposet of

$$P^n(B^\eta_1, \langle u_{b_1 \restriction \xi}\rangle^{Fi}) \times \cdots \times P^n(B^\eta_k, \langle u_{b_k \restriction \xi}\rangle^{Fi})$$

hence the countable set $X \restriction \xi$ will be dense in the smaller poset.

Obviously a lot of details have been omitted but let us include for illustration one of the interesting ideas for carrying out the induction mentioned above. If we set $A_0 = A_{b \restriction \xi}$ and $F = u_{b \restriction \xi}$ for any one of the $b_i$'s and assume that $X$ (for $X \restriction \xi$) is a maximal antichain in $P(A_0, F)$. Further assume that $B$ is a Boolean algebra containing $A_0$ such that $X$ remains maximal in $P(B, \langle F\rangle^{Fi})$, then we show that $X$ remains maximal in $P(B', \langle F\rangle^{Fi})$ where $B'$ is generated by $B \cup \{g\}$ where $g$ (like some $a_t$) is the generic added by $P(B, u)$ for some ultrafilter $u \supset F$ on $B$. Note that $F$ is an ultrafilter on $A_0$ but not on $B$.

The general element of $P(B', \langle F\langle^{Fi})$ has the form $r = ((g \cap e_1) \cup (f_1 - g) \ , \ (g \cap e_2) \cup (f_2 - g))$ for elements $e_1, f_1, e_2, f_2 \in B$. Now this is taking place in the extension by $g$, so that some $p \in P(B, u)$ forces that $r \in P(B', \langle F\rangle^{Fi})$ is incompatible with each member of $X$. Since $r_{-1} \cap r_1$ is empty, it follows that neither $e_1 \cap e_2$ nor $f_1 \cap f_2$ can be in $u$. In addition, $e_1 \cap f_1$ and $e_2 \cap f_2$ may be assumed not to be in $u$ since otherwise both $r_{-1}$ and $r_1$ can be shown to be in $B$. Therefore we may assume that $p \Vdash r = ((g \cap e_1) \cup f_1 \ , \ e_2 \cup (f_2 - g))$. Set $p^* = ((p_1 \cap e_1) \cup f_1 \ , \ e_2 \cup (p_{-1} \cap f_2))$ and check that $r^*$ is in $P(B, \langle F\rangle^{Fi})$. For example, $f_1 \cap (p_{-1} \cap f_2)$ is empty since $p \Vdash p_{-1} \cap f_2 \subset f_2 - g$. Choose an $x = (x_{-1}, x_1) \in X$ which is compatible with $r^*$ and set $p^* = (p_{-1} \cup (e_1 \cap x_1) \ , \ p_1 \cup (f_2 \cap x_{-1})) \in P(B, u)$. The fact that $r^*$ is compatible with $x$ ensures $p^*_{-1} \cap p^*_1 = \emptyset$ and $x \in P(B, \langle F\rangle^{Fi})$ ensures that $x_{-1} \cup x_1$ is disjoint from a member of $F$.

Finally, we check that $p^*$ forces that $x$ is compatible with $r$. All that needs to be checked is that $x_{-1}$ is disjoint from $r_1$ and $x_1$ is disjoint from $r_{-1}$. We check the first case.

$$x_{-1} \cap (e_2 \cup (f_1 - g)) \subset (x_{-1} \cap e_2) \cup ( \ (x_{-1} \cap f_2) - g) \ .$$

Now $x_{-1} \cap e_2$ is empty because $x$ is compatible with $r^*$ while the second term is empty since $p^* \Vdash x_{-1} \cap f_2 \subset g$.

Finally we turn to the result of Eisworth. This result uses the technique originating with Jensen's proof that CH is consistent with there being no Souslin trees, or perhaps more accurately with Shelah's general machinery for iterating certain proper forcings without adding reals [24]. The authors of [12] also attribute the paper of Avraham and Todorcevic [1] as a major influence. The chief consideration is to establish the extra conditions on the posets which will permit them to be iterated without adding reals. We refer the reader to either [11] or [12] for the innovation that expanded this class. We are content with examining the more basic properties of the single stage poset in this case.

**Theorem 6.10.** *It is consistent with CH that any locally compact countably compact first countable space which has compactification of countable tightness is already compact.*

In fact, if $\mathcal{U}$ is any maximal free filter of closed subsets of a space $X$ as in the theorem, then there is a poset $P$ which does not add reals and which forces there to be a copy of $\omega_1$ in the space $X$ which meets every member of $\mathcal{U}$. Note that by Lemma 3.3, we may assume that each member of $\mathcal{U}$ contains a separable member of $\mathcal{U}$. In the following, *large* means meets every member of $\mathcal{U}$, while *small* will mean not large.

Given a neighborhood assignment $f$ on a large subset, $\operatorname{dom} f$, of $X$ define $Ban(f) = \{y \in X : y \in f(x)$ for a small set of $x$'s $\}$. Using that $X$ is first countable and that $\mathcal{U}$ is countably complete, one can show that $Ban(f)$ is closed. Also, using the fact that $\mathcal{U}$ has a base of separable members, one can show that $Ban(f)$ is small. Such $f$ will be *promise*s in the conditions of the poset which is promising that the elements of the new copy of $\omega_1$ will not be coming from $Ban(f)$; in fact quite a bit more is promised.

**Definition 6.11.** Define a notion of forcing $P = P_X$ by putting $p$ into $P$ if and only if $p = ([p], \Phi_p)$ where

1. $[p]$ is a countable closed subset of $X$
2. $\Phi_p$ is a countable collection of promises.

A condition $q$ extends $p$ if $[q] \supset [p]$, $\Phi_q \supset \Phi_p$, for each $f \in \Phi_p$, the set

$$Y(f, q, p) = \{x \in \operatorname{dom} f : [q] \setminus [p] \subset f(x)\}$$

is large and $f \restriction Y(f, q, p) \in \Phi_q$.

One of the key steps to the proof is that given a dense set $D$ of $P$, and a condition $p$, let

$$Bad(p, D) = \{x \in X : x \text{ has a neighborhood } U_x \text{ such that there is no } q \leq p \text{ with } q \in D \text{ and } [q] \setminus [p] \subset U_x\} \tag{6.1}$$

then $Bad(p, D)$ is small.

To see this, assume that $Bad(p, D)$ is large. Let $f(x) = U_x$ for $x \in Bad(p, D)$ where $U_x$ is as in the definition of $Bad(p, D)$. It is trivial that $([p], \Phi_p \cup \{f\}) < p$ and so has an extension $q \in D$. By the definition of $<$, $Y(f, q, p)$ is large

We finish by showing that $P$ is proper and does not add reals. More specifically, given a countable $M \prec H(\lambda)$ with $X, \mathcal{U}$ and $P$ in $M$ and $p \in M \cap P$, there is a $q < p$ which such that $q \in D$ for each dense open $D \in M$. Let $\{D_n : n \in \omega\}$ list all dense open subsets of $P$ that are members of $M$. The following inductive construction is really the heart of the matter (after suitably defining the poset of course). Build $p_{n+1} \leq p_n$ by induction with $p_0 = p$ and $p_{n+1} \in M \cap D_n$ in such a way to ensure there is a lower bound. We must ensure that for each $m \in \omega$ and each promise $f \in \Phi_m$,

$$Y = \{y : \bigcup_{n \in \omega} [p_n] \setminus [p_m] \subset f(y)\} \text{ is large}$$

If so, we'll set $\Phi_q = \bigcup_{n\in\omega} \Phi_{p_n} \cup \{f \restriction Y : f \in \Phi_{p_n} n \in \omega\}$. Moreover to ensure that the set $\bigcup\{[p] : p \in G\}$ for a generic $G$ is homeomorphic to $\omega_1$, we have to ensure that there is a point $x$ such that $\{x\} \cup \bigcup_{n\in\omega}[p_n]$ is compact.

Let $\{x_n : n \in \omega\}$ be chosen so that it converges to some point $x$ and for each $U \in \mathcal{U} \cap M$, $\{x_n : n \in \omega\} \setminus U$ is finite. Check that if $f \in M$ is a promise, then $x \notin Banf$. In defining the sequence $p_n$ we also define a function $g \in \omega^\omega$. We also enumerate all the promises in $M$ because without loss the dense open sets of $M$ guarantee they will be in some $\Phi_m$. We can assume that promise $f_n$ comes from $\Phi_{p_m}$ for some $m \leq n$. Also let $U_n(x)$ enumerate a neighborhood base at $x$.

Set $B = Y(f_n, p_n, p_m) = \{y \in \operatorname{dom} f_n : [p_n] \setminus [p_m] \subset f_n(y)\}$. By defninition of extension, $B$ is large and $f_n \restriction B \in \Phi_{p_n}$. Since $x$ is not banned by $f_n \restriction B \in M$ we can find a neighborhood $V \subset U_n(x)$ of $x$ such that $K(V, f_n) = \{y \in B : V \subset f_n(y)\}$ is large. For ease of notation, assume that $V = U_n(x)$. Since $Bad(p_n, D_n) \cap \{x_n : n \in \omega\}$ is finite, we can choose $g(n)$ so that $x_{g(n)} \in V$ and there is an open neighborhood $V_{g(n)} \subset V$ of $x_{g(n)}$ which is contained in $V \setminus Bad(p_n, D_n)$. Now $V_{g(n)}$ is a neighborhood of $x_{g(n)} \notin B(p_n, D_n)$ so in $M$ we can find $p_{n+1} \leq p_n$, in $D_n$ such that $[p_{n+1}] \setminus [p_n] \subset V_{g(n)} \subset V$.

Let $B' = \{y \in \operatorname{dom} f_n : [p_n] \setminus [p_m] \subset f(y)\}$. Since $[q] \setminus [p_n] \subset V$, it follows that $[q] \setminus [p_m] \subset f(y)$ for enough $y$.

Finally, to ensure that the new set is uncountable one proves it as follows.

**Proposition 6.12.** *If $A$ is large, then there is a dense set of conditions $q$ with $[q] \cap A \neq \emptyset$.*

*Proof.* Fix any condition $p$ and recall that $Ban(f)$ is small for each $f \in \Phi_p$. Therefore there is a $U \in \mathcal{U}$ such that $U \cap Ban(f)$ is empty for each $f \in \Phi_p$. Since $A$ is large, there is an $a \in A \cap U$ and, of course, $Y(f, p) = \{x \in \operatorname{dom}(f) : a \in f(x)\}$ is large for each $f \in \Phi_p$. Set $[q] = [p] \cup \{a\}$ and $\Phi_q = \Phi_p \cup \{f \restriction Y(f, p) : f \in \Phi_p\}$. Then $q < p$ as required. □

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