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Two applications of reflection and forcing to topology

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Abstract A brief review of the method of reflection and forcing is given. This technique is then applied to show that two metrizability theorems following from CH are also consistent with the negation of CH (one is relative to the existence of a weakly compact cardinal). The first is "a compact space is metrizable if so are all of its subspaces of size $\aleph_1$". The second (requiring a weakly compact cardinal) is "a first countable compact space with a small diagonal is metrizable". The forcing for the first is simple Cohen real forcing and the second is the Mitchell collapse.

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Introduction. All spaces are Hausdorff. In recent years many important and interesting questions in general topology have been resolved using the method of reflection and forcing together with large cardinals. However, large cardinals are not an essential element of the technique, their only role is to allow one to assume more reflection. Indeed most forcing arguments involve some sort of reflection. In this article we begin with a brief and simplified exposition of this method. We then give two applications as mentioned in the abstract, one of which does not require a large cardinal but for which this technique seems particularly suitable. A much more detailed exposition of this technique (using supercompact cardinals and a slightly different approach) is available in [DTW].

Suppose that Γ is a class of topological objects; e.g. $\mathfrak{X} \in \Gamma$ implies $\mathfrak{X} = \langle X,\tau,\mathcal{C}\rangle$ where $\langle X,\tau\rangle$ is a topological space and $\mathcal{C}$, for instance, is a family of subsets of X. We shall say that $\mathcal{Y}$ is a subobject of $\mathfrak{X}$ if for some $Y \subset X$, $\mathcal{Y} = \langle Y,\tau|Y,\mathcal{C}'\rangle$ where, in this case, $\mathcal{C}' = \{C \cap Y : C \in \mathcal{C}\}$. (The demand on $\mathcal{C}'$ may vary but we do insist that $\mathcal{Y}$ is a subspace of $\mathfrak{X}$). We would say that $\mathcal{Y}$ has size less than κ if $|Y| < \kappa$ and $|\mathcal{C}'| < \kappa$.

Let us say that $\mathcal{R}(\Gamma,\kappa,\varphi)$ holds if and only if for every object $\mathfrak{X}$ in Γ, with $\varphi(\mathfrak{X})$ failing, $\mathfrak{X}$ has a subobject $\mathcal{Y}$ of size less than κ (not necessarily in Γ) for which $\varphi(\mathcal{Y})$ also fails. Equivalently, $\varphi(\mathcal{Y})$ holding for all subobjects $\mathcal{Y}$ of $\mathfrak{X}$ of size less than κ for any $\mathfrak{X} \in \Gamma$ implies $\varphi(\mathfrak{X})$ holds if and only if $\mathcal{R}(\Gamma,\kappa,\varphi)$ holds. It will be convenient to let $\tilde{\Gamma}(\kappa,\varphi)$ denote the subclass of Γ consisting of those $\mathfrak{X}$ in Γ for which $\varphi(\mathcal{Y})$ holds for all subobjects $\mathcal{Y}$ of $\mathfrak{X}$ of size less than κ.

Many "large" cardinals have formulations in terms of reflection schemes of a slightly different nature. Of course if κ is uncountable then the following holds: for any set X, formula φ and ordinal β (> rank of X) if $V_\beta \models \varphi(X)$, then there is an elementary submodel M of V_β such that $X \in M$, $|M| < \kappa$ and $M \models \varphi(X)$. We shall frequently assume without mention that V_β, hence M, is a model of any needed axioms of ZFC. One would have in mind that it may be possible to derive that $\varphi(X \cap M)$ is valid in V thus obtaining a reflection theorem of the sort postulated in the previous paragraph. This, of course, is not in general true but if additional second order properties are imposed on M (possibly increasing the minimum size of such an M) it may well be possible. For example, we may demand that M is closed under ω-sequences ($[M]^\omega \subset M$). More generally, one method of imposing second order properties on M is to demand that a certain Π^1_1 formula holds (as is done in weakly compact (with $\beta = \kappa$) or supercompact reflection (see [De] or [Ma])). A formula $\varphi(X_1,\ldots,X_n)$ is a Π^1_1 formula if it is of the form $\forall X_0\, \psi(X_0,\ldots,X_n)$ and

$\psi(X_0,\dots,X_n)$ is the usual kind of formula in the language $\{\in, X_0,\dots,X_n\}$ with variables $v_0, v_1, \dots$ and the X_i are unary predicate symbols. For $\{A_1,\dots,A_n\} \subset \mathcal{P}(M)$, we say that $\langle M,\in,A_1,\dots,A_n\rangle \models \varphi(A_1,\dots,A_n)$ if for all $A_0 \subset M$, $\langle M,\in,A_0,\dots,A_n\rangle \models \psi(A_0,\dots,A_n)$.

Another difficulty enters when we return to our problem of $\mathcal{R}(\Gamma,\varkappa,\varphi)$. We would like to deduce $\mathcal{R}(\Gamma,\varkappa,\varphi)$ from the fact that $\varkappa$ reflects φ in the sense of the above paragraph. That is, if $\mathfrak{X} \in \Gamma$ and we can show that for some suitable M with $M \models \varphi(\mathfrak{X})$ and $|M| < \varkappa$, we have in fact that $\varphi(\text{"}\mathfrak{X} \cap M\text{"})$ holds in V ("$\mathfrak{X} \cap M$" should be clear from context). However, for $\mathcal{R}(\Gamma,\varkappa,\varphi)$ we are demanding that "$\mathfrak{X} \cap M$" is a subobject of $\mathfrak{X}$. This is achieved by a combination of second order conditions on M (but not too many since we must have $|M| < \varkappa$) and local conditions on Γ. For instance it suffices to have that $\mathfrak{X} \in \tilde{\Gamma}(\varphi,\varkappa)$ implies that $\mathfrak{X}$ is first countable.

Finally, we throw forcing into the brew. Let Γ be a class of P-names for some poset P. Assume that $\varkappa$ is sufficiently large to satisfy the following reflection condition: if $\dot{\mathfrak{X}} \in \Gamma$ and $p \in P$ is such that $p \Vdash_P \neg\varphi(\dot{\mathfrak{X}})$ then there is an M as above such that $p \cap M \Vdash_{P\cap M} \neg\varphi(\text{"}\dot{\mathfrak{X}} \cap M\text{"})$ (we are already assuming appropriate second order properties on M and P to get from $M \models \Vdash_P \neg\varphi(\dot{\mathfrak{X}})$ to the above). We shall also have to impose conditions on M and P in order to get that P is forcing isomorphic to $P \cap M * (P/P\cap M)$, i.e. $P \cap M$ is completely embedded in P. Finally, in order to conclude, as we would like, that for some $q \le p$, $q \Vdash_{\underline{\underline{P}}} \neg\varphi(\text{"}\dot{\mathfrak{X}} \cap M\text{"})$, we must show that "$P/P\cap M$ preserves $\neg\varphi(\text{"}\dot{\mathfrak{X}} \cap M\text{"})$". We will also need to go back and check that the local conditions on Γ are sufficient to ensure that $\dot{\mathfrak{X}} \cap M$ is a subobject of $\dot{\mathfrak{X}}$; again first countability suffices but see [DTW] for a more general situation.

Let us now record a slightly more specific summary of the above discussion. Let Γ be a formula which describes the class of topological objects we are interested in. For example, $\Gamma(X,\tau,\mathcal{C},\theta)$ holds implies that (without loss of generality, X is an ordinal) $\tau \subset X \times \theta \times X$ codes a base for a topology (where $(x,\alpha,y) \in \tau$ if and only if y is in the αth neighbourhood of x) and $\mathcal{C} \subset X \times \theta$ codes some indexed family of subsets of X (i.e. $C_\alpha = \{x : (x,\alpha) \in \mathcal{C}\}$). Also let $\varphi(v_1,v_2,v_3,v_4)$ be a topological property and assume that both Γ and φ are Π^1_1 formulas. Let $\varkappa$ be a cardinal and let $P \subset V_\varkappa$ be a $\varkappa$-cc poset. Our aim is to prove $1 \Vdash_P \mathcal{R}(\Gamma,\varkappa,\varphi)$. Let X and θ be ordinals and let $\dot{\tau}$ and $\dot{\mathcal{C}}$ be P-names of subsets of $X \times \theta \times X$ and $X \times \theta$

respectively. Suppose that $p \Vdash_P \Gamma(X,\dot\tau,\dot{\mathscr{C}},\theta)$ and $\neg\,\varphi(X,\dot\tau,\dot{\mathscr{C}},\theta)$; in fact we shall ignore a technical point never entering topology (?) and assume that there is a β large enough so that for any P-names $\dot\rho$, $\dot{\mathscr{D}}$ of subsets of $X \times \theta \times X$ and $X \times \theta$ respectively, $q \Vdash_P \Gamma(X,\dot\rho,\dot{\mathscr{D}},\theta)$ and $q \Vdash_P \varphi(X,\dot\rho,\dot{\mathscr{D}},\theta)$ hold if and only if they hold in V_β (see [DTW] for a discussion of local properties).

<u>Step 1</u> (Reflection) Let χ be a suitable Π^1_1 formula and $A \subset V_\beta$. Find (if possible) $M \prec V_\beta$ with $|M| < \kappa$ so that (in the Π^1_1 sense) $\langle M,\in,P,X,\dot\tau,\dot{\mathscr{C}},\theta,A\rangle \models [\chi(A)$ and $p \Vdash_P \Gamma(X,\dot\tau,\dot{\mathscr{C}},\theta) \wedge \neg\,\varphi(X,\dot\tau,\dot{\mathscr{C}},\theta)]$.

<u>Step 2</u> (Absoluteness) Using χ and A deduce that
$p \cap M \Vdash_{P\cap M} \Gamma(X \cap M , \dot\tau \cap M , \dot{\mathscr{C}} \cap M , \theta \cap M) \wedge \neg\,\varphi(X , \dot\tau \cap M , \dot{\mathscr{C}} \cap M , \theta \cap M)$
and $P \cap M$ is completely embedded in P . (For example if P is Cohen real forcing, then $P \cap M$ is always completely embedded in P) .

<u>Step 3</u> (Preservation) Show that if G is $P \cap M$-generic over V then $V[G] \models$ forcing with $P/P\cap M$ preserves the failure of $\varphi(Y,\rho,\mathscr{D},\alpha)$ for spaces such that $\Gamma(Y,\rho,\mathscr{D},\alpha)$ holds. Deduce that $p \cap M \Vdash_P \neg\,\varphi(X \cap M , \dot\tau \cap M , \dot{\mathscr{C}} \cap M , \theta \cap M)$.

<u>Step 4</u> (More absoluteness) Show that $p \cap M \Vdash_P \langle X \cap M , \dot\tau \cap M\rangle$ is a subspace of $\langle X,\dot\tau\rangle$.

It is sometimes necessary (especially for Steps 3 and 4) to work with $\tilde\Gamma(\kappa,\varphi)$ rather than Γ . However, in this case we would expect to reach a contradiction since there should be no spaces in $\tilde\Gamma(\kappa,\varphi)$ for which φ fails. Frequently the contradiction is simply obtained by showing $\neg\,\varphi(\text{"}\dot{\mathfrak{X}} \cap M\text{"})$ (since we are assuming $\varphi(\text{"}\dot{\mathfrak{X}} \cap M\text{"})$) .

<u>Proposition</u>. Assume that $P \cap M$ is completely embedded in P and that $1 \Vdash_P \langle X,\dot\tau\rangle$ is first countable (in fact assume $1 \Vdash_P \dot\tau \subset X \times \omega \times X$) . Then step 4 holds; that is $1 \Vdash_P \langle X \cap M , \dot\tau \cap M\rangle$ is a subspace of $\langle X,\dot\tau\rangle$.

<u>Proof</u>. Since $P \cap M$ is completely embedded in P we may assume that $P = P \cap M * \dot Q$ for some $\dot Q$. Let $x,y \in X \cap M$ and $n \in \omega$ and assume that $\langle p,\dot q\rangle \Vdash \langle x,n,y\rangle \in \dot\tau$. It suffices to prove that $\langle p,1\rangle \Vdash_P \langle x,n,y\rangle \in \dot\tau$. For

all $p' \le p$ in M, $\langle p',1\rangle$ is compatible with $\langle p,\dot{q}\rangle$, hence it cannot be the case that $p' \Vdash_{P\cap M} \langle x,n,y\rangle \notin \dot{\tau}$. Therefore, by absoluteness, $M \models \neg\, p' \Vdash_P \langle x,n,y\rangle \notin \dot{\tau}$ for all $p' \le p$. In other words, $M \models p \Vdash_P \langle x,n,y\rangle \in \dot{\tau}$ from which it follows that $p \Vdash_P \langle x,n,y\rangle \in \dot{\tau}$.

A situation discussed at length in [DTW] is the case when κ is supercompact (or κ is weakly compact and each of X,θ and β are at most κ). In this case we can formulate χ and A in order to guarantee that there is a strongly inaccessible $\lambda < \kappa$ so that $M \cap V_\kappa = V_\lambda$, $[M]^{<\lambda} \subset M$ and $P \cap M$ has the λ - cc (in V). Now step 1 can be accomplished by the properties of κ being supercompact (or weakly compact) (see [Ma], [Ka Ma], [De] or [DTW]). Using χ we have already guaranteed that $P \cap M$ is completely embedded in P and it is quite likely that all of step 2 holds, but this must be checked (also χ can be further strengthened). In most cases, Step 3 is the heart of the argument and is where the greatest difficulty lies. Step 4 can be shown to hold, for example, if each point of $\mathfrak{X}$ has character less than κ (again see [DTW]).

In section 4 we present an argument where it is sufficient to have that $[M]^\omega \subset M$ hence M can be chosen to have cardinality c. In this case we are using Cohen real forcing, hence $P \cap M$ will be completely embedded in P, but Steps 3 and 4 requires some work.

2. The forcing posets.

In this section we review the basic facts of the two forcing notions we shall use. The first and simplest class of forcings are the well known Cohen real posets denoted $Fn(I,2)$ where I is any set. Following Kunen [K], $Fn(I,2) = \{p : p$ is a function, $dom(p) \in [I]^{<\omega}$ and $range(p) \subset 2\}$ and is ordered by reverse inclusion. Recall that if J is any subset of I then $Fn(I,2)$ is forcing isomorphic to $Fn(J,2) * Fn(I - J,2)$. If $\dot{X}$ is a $Fn(I,2)$-name and G is $Fn(I,2)$-generic, we shall let $val(\dot{X}, G \cap Fn(J,2))$ denote the unique nice $Fn(I - J,2)$-name in $V[G \cap Fn(J,2)]$ such that in $V[G]$ $(= V[G \cap Fn(I,2)][G \cap Fn(I - J,2])$ $val(\dot{X},G) = val(val(\dot{X}, G \cap Fn(J,2)), G \cap Fn(I - J,2))$.

The other forcing notion which we require is the "Mitchell collapse".

2.1 Definition. For an ordinal θ, let $Mi(\theta)$ denote the iteration $\langle P_\alpha, \dot{Q}_\alpha\rangle_{\alpha<\theta}$ where for α even $\dot{Q}_\alpha = Fn(\omega_2,2)$ and α odd $\dot{Q}_\alpha = {}^{<\omega_1}2$ ordered by reverse inclusion. The ideal of supports for $Mi(\theta)$ is generated by the set of finite subsets of the even ordinals union the set of countable subsets of the odds.

2.2 Basic Facts about $Mi(\theta)$.

(i) If G is $Mi(\theta)$-generic, for θ strongly inaccessible, then $V[G] \models \aleph_1 = \check{\omega}_1$, $2^{\aleph_0} = \check{\theta} = \aleph_2$.

(ii) If $\lambda < \theta$ is even, then $Mi(\theta)$ is forcing isomorphic to $Mi(\lambda) * Mi(\theta')$ where $\lambda + \theta' = \theta$.

(iii) There is a $Mi(\theta)$-name, $\dot{R}$, of a poset such that $Mi(\theta) * \dot{R}$ is forcing isomorphic to $Fn(\mu,2) \times Q$ for some ω_1-closed poset Q and some uncountable μ .

(iv) There is a $Mi(\theta)$-name of a poset $\dot{R}$ and a $Fn(\omega_1,2)$-name of a poset $\dot{Q}$ such that $M(\theta) * \dot{R}$ is forcing isomorphic to $Fn(\omega_1,2) * (\dot{Q} \times Fn(\mu - \omega_1,2))$ such that $1 \Vdash_{Fn(\omega_1,2)} \dot{Q}$ is ω_1-closed.

Proof. (i) - (iii) are from Mitchell's original paper [Mi] and (iv) is an easy consequence of (iii) and the fact that $Mi(\theta)$ is forcing isomorphic to $Fn(\omega_1,2) * Mi(\theta)$.

2.3 Remark. If θ has uncountable cofinality we will view $Mi(\theta)$ as the union of the posets $\{Mi(\alpha) : \alpha < \theta\}$ rather than the usual notion of iteration in which the elements are functions with domain θ . This point of view is equivalent (as forcing notions) and allows us to view $Mi(\theta)$ as a subset of V_θ rather than $V_{\theta+1}$. Furthermore, if $\lambda < \theta$ is strongly inaccessible then $Mi(\theta) \cap V_\lambda$ is equal to $Mi(\lambda)$ and is completely embedded in $Mi(\theta)$.

One final general fact about both Cohen real forcing and Mitchell forcing is that they are both proper. The consequence of this fact which we wish to record is the following.

2.4 Proposition. If P is either $Fn(\theta,2)$ or $Mi(\theta)$ for any θ and $\dot{x}$ is a P-name such that $p \Vdash \dot{x} \in \mu$ for any $p \in P$ and ordinal μ , then there is a countable $A \subset \mu$ and a $q \le p$ such that $q \Vdash \dot{x} \in \check{A}$. In the case P is $Fn(\theta,2)$, q can be chosen to be p .

Proof. This follows easily from 2.2 (iii) and the fact that $Fn(\theta,2)$ is ccc.

3. The preservation lemmas.

In this section we shall prove the preservation properties which we shall need to complete Step 3 in our outline.

3.1 Lemma. Let $\langle X,\tau\rangle$ be a space with the property that each uncountable subset of X has a countable subset with a limit point. If P is $\mathrm{Fn}(\mu,2)$ (for any μ) or if P is ω_1-closed then $1 \Vdash_P$ each uncountable $Y \subset \check{X}$ has a countable subset with a limit point in $\langle X,\tau\rangle$.

Proof. The simplest case is when P is ω_1-closed. Indeed let $\dot{A}$ be a P-name of a one-to-one function from ω_1 into X. Let $p_0 \in P$ be arbitrary and choose recursively a descending sequence $\{p_\alpha : \alpha < \omega_1\} \subset P$ and a sequence $\{a_\alpha : \alpha < \omega_1\} \subset X$ such that, for all $\alpha < \omega_1$, $p_{\alpha+1} \Vdash \dot{A}(\alpha) = a_\alpha$. Now by assumption there is a $\beta < \omega_1$ such that $\{a_\alpha : \alpha < \beta\}$ has an accumulation point. Clearly $p_{\beta+1} \Vdash \{\dot{A}(\alpha) : \alpha < \beta\}$ has an accumulation point. Now, suppose $P = \mathrm{Fn}(\mu,2)$ and again let $\dot{A}$ be a P-name as above and q any element of P. For each $\alpha < \omega_1$, choose $p_\alpha \in P$, $p_\alpha < q$, and $a_\alpha \in X$ such that $p_\alpha \Vdash \dot{A}(\alpha) = a_\alpha$. Let $p \in P$ and $I \in [\omega_1]^{\omega_1}$ be chosen so that $\{p_\alpha : \alpha \in I\}$ forms a Δ-system with root $p \supset q$. It is routine to check that for any $\beta < \omega_1$ and accumulation point x of $\{a_\alpha : \alpha \in I \cap \beta\}$, $p \Vdash x$ is an accumulation point of $\{\dot{A}(\alpha) : \alpha \in I \cap \beta\}$.

3.2 Lemma. Let $\mu \geq \omega_1$ and $1 \Vdash_{\mathrm{Fn}(\mu,2)} \dot{Q}$ is ω_1-closed. Let M be a countable elementary submodel of V_β for a sufficiently large β so that $\mathrm{Fn}(\mu,2) * \dot{Q} \in M$. There is a $\mathrm{Fn}(\mu,2)$-name $\dot{q}_0$ such that $1 \Vdash \dot{q}_0 \in \dot{Q}$ and for any $\mathrm{Fn}(\mu,2) * \dot{Q}$-generic G over V with $\langle 1,\dot{q}_0\rangle \in G$, the set $\{\langle p,\dot{q}\rangle \in G \cap M : p \Vdash \dot{q}_0 < \dot{q}\}$ is $M \cap \mathrm{Fn}(\mu,2) * \dot{Q}$-generic over V.

Proof. It is easily checked that there is a $\mathrm{Fn}(\mu \cap M,2)$-name $\dot{Q}'$ such that $1 \Vdash \dot{Q}'$ is a countable atomless subposet of $\dot{Q}$ and $M \cap (\mathrm{Fn}(\mu,2) * \dot{Q})$ is isomorphic to $\mathrm{Fn}(\mu \cap M,2) * \dot{Q}'$. Since M is only countable, we can find a $\mathrm{Fn}(\mu,2)$-name $\dot{F}$ such that for any $\mathrm{Fn}(\mu \cap M,2)$-generic $\tilde{G}$ over V, $\mathrm{val}(\dot{F},\tilde{G})$ is $\mathrm{val}(\dot{Q}',\tilde{G})$-generic over $V[\tilde{G}]$ (where $\mathrm{val}(\dot{F},\tilde{G})$ is the usual $\mathrm{Fn}(\mu - M,2)$-name for $\dot{F}$). Finally, since $1 \Vdash_{\mathrm{Fn}(\mu,2)} \dot{Q}$ is ω_1-closed, we choose $\dot{q}_0$ to be a lower bound to $\dot{F}$.

3.3 Lemma. Let $\mu \geq \omega_1$, and let $\dot{Q}$ be a $\mathrm{Fn}(\mu,2)$-name such that $1 \Vdash_{\mathrm{Fn}(\mu,2)} \dot{Q}$ is ω_1-closed. If $\langle X,\tau\rangle$ is a Lindelöf space, then $1 \Vdash_{\mathrm{Fn}(\mu,2)*Q} \langle X,\tau\rangle$ is Lindelöf.

Proof. Let $\dot{\mathcal{U}}$ be a $\mathrm{Fn}(\mu,2) * \dot{Q}$-name and assume that $\langle p,\dot{q}\rangle \Vdash \dot{\mathcal{U}} \subset \check{\tau}$ is an open cover of $\check{X}$. Let M be a countable elementary submodel of a large enough V_β and assume $\{\dot{\mathcal{U}},\langle p,\dot{q}\rangle\} \subset M$. Choose $\dot{q}_0$ as in 3.2 such that, in addition, $p \Vdash \dot{q}_0 < \dot{q}$. We claim $\langle p,\dot{q}_0\rangle \Vdash \dot{\mathcal{U}} \cap M$ covers $\check{X}$. By 3.2 it suffices to prove that for each $x \in X$, $\mathcal{D}_x = \{\langle p_1,\dot{q}_1\rangle \in M \cap \mathrm{Fn}(\mu,2) * \dot{Q} :$ there is a $U \in \tau \cap M$ with $x \in U$ and $\langle p_1,\dot{q}_1\rangle \Vdash \check{U} \in \dot{\mathcal{U}}\}$ is $M \cap (\mathrm{Fn}(\mu,2) * \dot{Q})$-dense. So, let $\langle r,\dot{s}\rangle \in M \cap (\mathrm{Fn}(\mu,2) * \dot{Q})$ be arbitrary. Since $\langle X,\tau\rangle$ is Lindelöf, and $\langle r,\dot{s}\rangle \Vdash \bigcup \dot{\mathcal{U}} = \check{X}$, there are sequences $\{U_n : n \in \omega\} \subset \tau$ and $\{\langle r_n,\dot{s}_n\rangle : n \in \omega\} \subset \{\langle r',\dot{s}'\rangle \in \mathrm{Fn}(\mu,2) * \dot{Q} : \langle r',\dot{s}'\rangle < \langle r,\dot{s}\rangle\}$ such that $\bigcup_n U_n = X$ and $\langle r_n,\dot{s}_n\rangle \Vdash \check{U}_n \in \dot{\mathcal{U}}$ for each $n \in \omega$. Therefore, by elementarity, this is true in M . Furthermore, $\{U_n : n \in \omega\} \in M$ and $\{\langle r_n,\dot{s}_n\rangle : n \in \omega\} \in M$ imply $\{U_n : n \in \omega\} \cup \{\langle r_n,\dot{s}_n\rangle : n \in \omega\} \subset M$. Finally, $\bigcup_n U_n = X$ implies $\langle r_n,\dot{s}_n\rangle \in \mathcal{D}_x$ for some $n \in \omega$.

For a space X , a subset F of X and an indexed sequence $\{a_\alpha : \alpha < \mu\} \subset X$, let us say that $\{a_\alpha : \alpha < \mu\}$ converges to F if every neighbourhood of F contains $\{a_\alpha : \beta < \alpha < \mu\}$ for some $\beta < \mu$.

3.4 Lemma. Suppose X is a space and $\{a_\alpha : \alpha < \mu\} \subset X - F$ converges to $F \subset X$ where μ has uncountable cofinality. If G is $\mathrm{Fn}(\theta,2)$-generic over V , for any θ , then $\{a_\alpha : \alpha < \mu\}$ still converges to F in $V[G]$.

Proof. It suffices to show that if $\dot{I}$ is a $\mathrm{Fn}(\theta,2)$-name and $p \in \mathrm{Fn}(\theta,2)$ with $p \Vdash \dot{I} \subset \mu$ is cofinal, then $q \Vdash \{a_\alpha : \alpha \in \dot{I}\}$ has a limit point in F , for some $q < p$. Now, for each $\alpha < \mu$, choose, if possible, $p_\alpha \leq p$ such that $p_\alpha \Vdash \alpha \in \dot{I}$. Let $J \subset \mu$ be cofinal with minimum cardinality such that p_α exists for each $\alpha \in J$. Since $|J|$ is an uncountable regular cardinal, there is a subset $J' \subset J$ with $|J'| = |J|$ such that $\{p_\alpha : \alpha \in J'\}$ forms a Δ-system with roof q . Let $x \in F$ be a limit point of $\{a_\alpha : \alpha \in J'\}$ (since

J' is cofinal in μ). Since q is the root of the Δ-system $\{p_\alpha : \alpha \in J'\}$, $\{\alpha \in J' : q' \Vdash \alpha \notin \dot{I}\}$ is finite for any $q' < q$. Therefore $q \Vdash x$ is a limit point of $\{a_\alpha : \alpha \in \dot{I}\}$.

4. $\mathfrak{R}(\Gamma, \aleph_2$, <u>metrizable</u>)

Recall that a space X has tightness at most κ if for any $x \in X$ and $A \subset X$ with x a limit point of A there is a $B \in [A]^{\leq\kappa}$ such that x is a limit point of B. Let "$t \leq \kappa$" denote the class of regular spaces having tightness at most κ and let "$\chi = \aleph_0$" denote the class of first countable spaces.

The consistency of $\mathfrak{R}(t \leq \aleph_1, \aleph_2$, metrizable) (relative to a large cardinal) is still very much open and seems very similar to the problem of producing a model with "$\underline{c} = \aleph_2$ and every normal Moore space is metrizable". It is shown in [DTW] that $\mathfrak{R}(t \leq \aleph_1, \underline{c}$, metrizable) is consistent relative to a strongly compact cardinal. However, in this section we intend to demonstrate the consistency with "$c \geq \aleph_2$" of $\mathfrak{R}$(compact, $\aleph_2$, metrizable). We shall also consider the related problems of $\mathfrak{R}(\Gamma, \aleph_2$, metrizable) where Γ is one of "$\aleph_0$-compact", "$\aleph_1$-compact" and "Lindelöf" (recall that X is κ-compact if every subset of X of cardinality at least κ has a limit point). Observe that some assumption on Γ is necessary since obviously $\mathfrak{R}$(normal Hausdorff, κ, metrizable) fails for all κ.

To begin, let us first show that $\mathfrak{R}(\aleph_1$-compact, $\underline{c}^+$, metrizable) holds (Juhasz [J] observes this for Γ = "compact"). Note that if a space X is any of compact, $\aleph_0$-compact or Lindelöf then X is $\aleph_1$-compact. We shall give a proof which we hope will aid in understanding the method of reflection rather than the simplest proof which is to utilize Hajnal and Juhasz's result that $\mathfrak{R}$(Top, $\aleph_2$, countable weight) holds where Top is the class of all topological spaces. As discussed earlier, let X be a set and let $\tau \subset X \times \theta \times X$ code the topology on X so that $\langle X, \tau \rangle \in \tilde{\Gamma}$ ($\underline{c}^+$, metrizable) where Γ is the class of $\aleph_1$-compact spaces. Let β be large enough and let M be an elementary submodel of V_β so that $\langle X, \tau \rangle \in M$, $|M| \leq \underline{c}$ and $[M]^\omega \subset M$ (this can be done by building M as the union of an elementary chain of length ω_1). Let $Y = M \cap X$ and let $\tau' = T \cap M$. We shall proceed by proving a series of facts.

<u>Fact 1</u>: $\langle Y, \tau|_Y \rangle$ is $\aleph_1$-compact.

Proof. Let $A \in [Y]^{\omega_1}$ and choose $x \in X$ such that x is a limit point of A. Since $A \cup \{x\}$ is metrizable as a subspace of X, there is an $A' \in [A]^{\omega}$ which converges to x. Therefore $A' \in M$ (since $[M]^{\omega} \subset M$) and in fact $x \in M$ since $M \models \exists x$, x is the unique limit of A'.

Fact 2: $\tau \cap M = \tau|_Y$ (this is Step 4 of our method).

Proof. Suppose A is a $\tau|_Y$-closed subset of Y. Since A is an $\aleph_1$-compact metrizable subspace of X (recall that $X \in \tilde{\Gamma}(\underline{c}^+, \text{metrizable})$ and $|A| \le \underline{c}$), A is separable. Let $A' \in [A]^{\omega} \cap M$ be τ-dense in A. Now since $A' \in M$ the formula "x is in the τ-closure of A'" is absolute for M, hence A is $\tau \cap M$-closed as well.

Finally we complete the proof by showing that Step 1 of our method holds.

Fact 3. $\langle Y, \tau \cap M\rangle$ is not metrizable.

Proof. By Fact 1, $\langle Y, \tau \cap M\rangle$ is $\aleph_1$-compact since $\tau \cap M$ is always a sub-topology of $\tau|_Y$ (in fact the same, by Fact 2). Therefore, if this space were metrizable it would have a countable base. By decoding our code τ, we would have a countable $A \subset Y = X \cap M$ and a countable $B \subset \theta \cap M$ so that the set $\{\{y \in X \cap M : \langle x,\alpha,y\rangle \in \tau \cap M\} : x \in A, \alpha \in B\}$ is a base for $\langle Y, \tau \cap M\rangle$. However, both A and B are elements of M since $[M]^{\omega} \subset M$ hence by absoluteness $\{\{y \in X : \langle x,\alpha,y\rangle \in \tau\} : x \in A, \alpha \in B\}$ really is a base for X contradicting that X is not metrizable.

Corollary CH implies that $\mathcal{R}(\aleph_1\text{-compact}, \aleph_2, \text{metrizable})$ holds.

We shall now show, with a very similar argument, the main result of this section.

Theorem 4.1. If V is a model of CH and G is $Fn(\mu,2)$-generic over V (for any μ) then $\mathcal{R}(\aleph_1\text{-compact}, \aleph_2, \text{metrizable})$ holds in $V[G]$.

Corollary 4.2. Each of $\mathcal{R}(\text{compact}, \aleph_2, \text{metrizable})$, $\mathcal{R}(\aleph_0\text{-compact}, \aleph_2, \text{metrizable})$ and $\mathcal{R}(\text{Lindelöf}, \aleph_2, \text{metrizable})$ is consistent with the negation of CH.

Proof of Theorem 4.1. Let Γ be the class of $\aleph_1$-compact Hausdorff spaces and let X be any cardinal. Suppose that $\dot{\tau}$ is an $Fn(\mu,2)$-name such that $1 \Vdash \langle X,\dot{\tau}\rangle \in \tilde{\Gamma}$ ($\aleph_2$, metrizable) and $\langle X,\dot{\tau}\rangle$ is not metrizable. As above, let M be an elementary submodel of a sufficiently large V_β so that $|M| = \aleph_1$, $[M]^\omega \subset M$ and everything relevant is in M. We again proceed with a series of Facts.

Fact 4. $1 \Vdash_{Fn(\mu\cap M,2)}$ "for any uncountable $Y \subset X \cap M$, Y has a countable subset with a limit point in $\langle X \cap M , \dot{\tau} \cap M\rangle$".

Proof. Let $\dot{A}$ be a nice $Fn(\mu \cap M,2)$-name such that $1 \Vdash \dot{A}$ is an uncountable subset of $X \cap M$. Since $Fn(\mu,2)$ is a ccc forcing, we can find $\{a_\alpha: \alpha < \omega_1\}$ such that $1 \Vdash \dot{A} \subset \{a_\alpha: \alpha < \omega_1\}$. Furthermore, since $1 \Vdash \langle X,\dot{\tau}\rangle \in \tilde{\Gamma}$ ($\aleph_2$, metrizable) (and using 2.4) there is an $\alpha < \omega_1$ such that $1 \Vdash \dot{A} \cap \{a_\gamma: \gamma < \alpha\}$ has a limit point. Now, in fact, there is a countable name $\dot{B} \subset M$ such that $1 \Vdash \dot{B} = \dot{A} \cap \{a_\gamma: \gamma < \alpha\}$ (since $\dot{B} = \dot{A} \cap \{a_\gamma: \gamma < \alpha\} \times Fn(\mu \cap M,2)$, see [K] for details on nice names). Therefore, $\dot{B} \in M$ and by absoluteness "$1 \Vdash \dot{B}$ has a limit point" holds in M.

Fact 5. $1 \Vdash_{Fn(\mu\cap M,2)} \langle X \cap M , \dot{\tau} \cap M\rangle$ is not metrizable.

Proof. This is basically the same as Fact 3. Indeed, by Fact 4, if it were metrizable, it would have a countable name for a base. This name would be in M and could not be a base by absoluteness.

Let us review what remains to be shown. In the model $V[G \cap M]$, we have that $\langle X \cap M , val(\dot{\tau} \cap M , G \cap M)\rangle$ is an $\aleph_1$-compact non-metrizable space. We must show (Step 3) that forcing with $Fn(\mu - M,2)$ preserves that it is non-metrizable and (Step 4) that, in $V[G]$, $\langle X \cap M , val(\dot{\tau} \cap M , G \cap M)\rangle$ is a subspace of $\langle X , val(\dot{\tau},G)\rangle$. It has been shown in [DTW] that $Fn(I,2)$ preserves "X is not metrizable" for any set I and space X. However, in this case, since we have so little Π^1_1 reflection (just $[M]^\omega \subset M$) and we are not assuming that X is first countable, we have to work hard for Step 4 and in doing so we can give an alternate proof for Step 3 as well.

Fact 6. In $V[G]$, for any countable $A \subset X \cap M$ the topologies on A generated by $val(\dot{\tau},G)$ and $val(\dot{\tau} \cap M, G \cap M)$ are the same.

Proof. By 2.4, we can asume that A is in V, hence in M (since $[M]^\omega \subset M$). Since $\langle X, val(\dot{\tau},G)\rangle \in \tilde{\Gamma}$, $val(\dot{\tau},G)|A$ has a countable base. Therefore, there is a countable name for this base in M which implies that $val(\dot{\tau},G)|_A \subset val(\dot{\tau} \cap M, G \cap M)|A$.

Fact 7. In $V[G]$, $\langle X \cap M, val(\dot{\tau},G)|_{X\cap M}\rangle$ is $\aleph_1$-compact.

Proof. By 3.1 and Fact 4, $\langle X \cap M, val(\dot{\tau} \cap M, G \cap M)\rangle$ has the property, in $V[G]$, that every uncountable subset of $X \cap M$ has a countable subset with a $val(\dot{\tau} \cap M, G \cap M)$-limit point in M. By Fact 6, this point is a $val(\dot{\tau},G)$-limit point of this same countable set.

Now, similar to Fact 2, we can prove:

Fact 8. In $V[G]$, $\langle X \cap M, val(\dot{\tau} \cap M, G \cap M)\rangle$ is a subspace of $\langle X, val(\dot{\tau},G)\rangle$.

Proof. Let $A \subset X \cap M$ be $val(\dot{\tau},G)$-relatively closed in $X \cap M$. Since we are assuming that $(X, val(\dot{\tau},G)) \in \tilde{\Gamma}$, $val(\dot{\tau},G) \mid X \cap M$ is metrizable. Furthermore, by Fact 7, A is $\aleph_1$-compact, hence there is a countable $D \subset A$ which is $val(\dot{\tau},G)$-dense in A (and therefore $val(\dot{\tau} \cap M, G \cap M)$-dense in A). For any $x \in X \cap M$, the two topologies agree on $\{x\} \cup D$ by Fact 6. Hence, A is $val(\dot{\tau} \cap M, G \cap M)$-closed.

We conclude the proof with:

Fact 9. In $V[G]$, $\langle X \cap M, val(\dot{\tau},G)|_{X\cap M}\rangle$ is not metrizable.

Proof. By 2.4 and Facts 5 and 8, $val(\dot{\tau},G)_{X\cap M}$ does not have a countable base. Therefore it is not metrizable since, by Fact 7, it is $\aleph_1$-compact.

With a very similar proof we can also show the following. Recall that a cardinal is Mahlo if it is strongly inaccessible and it has a stationary subset of strongly inaccessibles below it.

Theorem 4.3. If θ is Mahlo and G is $Mi(\theta)$-generic over V then $\mathcal{R}(\aleph_1\text{-compact}, \aleph_2, \text{metrizable})$ holds in $V[G]$.

Proof. Recall from §2, that, in $V[G]$, $\underline{c} = \aleph_2 = \theta$ and, from this section, that $\mathcal{R}(\aleph_1\text{-compact}, \underline{c}^+, \text{metrizable})$ holds. Therefore, to prove the theorem it suffices to prove that $\mathcal{R}(\aleph_1\text{-compact}, \aleph_2, \text{metrizable})$ holds for each topology on θ. Let $\tau \subset \theta \times 2^\theta \times \theta$ code a topology on θ such that $\langle\theta,\tau\rangle \in \tilde{\Gamma}(\aleph_2$, metrizable) where Γ is the class of $\aleph_1$-compact spaces. We can assume that for all $\lambda < \theta$ and $\alpha < \lambda$ the set $\{\{\beta : \langle\alpha,\lambda + n, \beta\rangle \in \tau\} : n \in \omega\}$ is a neighbourhood base for α in the subspace $\langle\lambda, \tau|\lambda\rangle$. Let $\dot\tau$ be a $Mi(\theta)$-name for τ. Since θ is strongly inaccessible, there is a continuous increasing function $g \in {}^\theta\theta$ such that, for each $\alpha \in \theta$, $\dot\tau \cap \alpha \times (\alpha+\omega) \times \alpha$ is a $Mi(g(\alpha))$-name, $1 \Vdash_{Mi(\theta)} A \times A \times g(\alpha)$ does not contain a base for $g(\alpha)$ for each countable $A \subset \alpha$ and for each $Mi(\alpha)$-name $\dot Y$ of an $\aleph_1$-sized subset of α $1 \Vdash_{Mi(\theta)} \dot Y$ has a countable subset with a limit point less than $g(\alpha)$. Since θ is Mahlo, g has a strongly inaccessible fixed point $\lambda < \theta$. Therefore $Mi(\lambda)$ has the λ - cc and for each $Mi(\lambda)$-name with $1 \Vdash \dot Y \in [\lambda]^{\omega_1}$, we can assume there is an $\alpha < \lambda$ such that $\dot Y$ is an $Mi(\alpha)$-name and $1 \Vdash \dot Y \in [\alpha]^{\omega_1}$. Since $g(\alpha) < \lambda$ for $\alpha < \lambda$, it follows that $1 \Vdash_{Mi(\lambda)}$ each uncountable subset of λ has a countable subset with a $\dot\tau \cap V_\lambda$-limit point in λ. By 3.1, $1 \Vdash_{Mi(\theta)}$ each uncountable subset of λ has a countable subset with a $\dot\tau \cap V_\lambda$-limit point in λ. Also, for each $\alpha < \lambda$, $\tau \cap \alpha \times (\alpha+\omega) \times \alpha$ contains a base for $\tau|\alpha$ and $\dot\tau \cap \alpha \times (\alpha+\omega) \times \alpha$ is an $Mi(g(\alpha))$-name, hence, in $V[G]$, $\langle\alpha, \text{val}(\dot\tau \cap V_\lambda, G \cap Mi(\lambda))\rangle$ is a subspace of $\langle\theta,\tau\rangle$. It follows that in $V[G]$, $\langle\lambda, \tau|\lambda\rangle$ is $\aleph_1$-compact. Again, since $\langle\lambda, \tau|\lambda\rangle$ is metrizable, every subset has a countable dense subset. Hence, for any $Y \subset \lambda$, there is an $\alpha < \lambda$ such that $Y \cap \alpha$ is dense in Y. For each $\beta < \lambda$, the neighbourhood trace of β on $Y \cup \{\beta\}$ is determined by its trace on $\alpha \cup \{\beta\}$. Therefore, by the first condition on g, $\langle\lambda, \text{val}(\dot\tau \cap V_\lambda, G)\rangle$ is a subspace of $\langle\theta,\tau\rangle$. However, by the second condition on g, no countable subset of $\dot\tau \cap V_\lambda$ codes a base for λ which contradicts that $\langle\lambda, \tau|\lambda\rangle$ is metrizable and $\aleph_1$-compact.

In contrast to the above two theorems we have:

Proposition 4.4. $MA(\omega_1)$ implies that $\mathcal{R}(\aleph_1\text{-compact}, \aleph_2, \text{metrizable})$ and $\mathcal{R}(\text{Lindelöf}, \aleph_2, \text{metrizable})$ fail.

Proof. Let Y be any subset of the unit interval I of cardinality $\aleph_1$. Let $X = I - Y \times \{0\} \cup Y \times \{1\}$ be a subspace of the usual Alexandroff double $I \times 2$ where $I \times \{1\}$ are the isolated points. It is clear that X is Lindelöf and not metrizable. Furthermore, by $MA(\omega_1)$, for any $Z \in [I]^{\leq\omega_1}$, $Z - Y \times \{0\} \cup Z \cap Y \times \{1\}$ is metrizable (each subset of Z is a relative G_δ, see [M]).

5. Small diagonal can imply metrizable.

Recall that a space X is said to have a small diagonal if for each uncountable $Y \subset X^2 - \Delta$ (where $\Delta = \{(x,x) \in X^2 : x \in X\}$) there is a neighbourhood U of Δ such that $Y \setminus U$ is uncountable. M. Husek proved that if CH holds, then each compact space with countable tightness and small diagonal is metrizable. H.-X. Zhou [Z] proved that the tightness assumption can be dropped (but he still used CH) if homeomorphic copies of the ordinal space ω_1 in first countable spaces always have a neighbourhood base of cardinality $\aleph_1$. W.G. Fleissner had proven earlier that this latter condition was consistent with CH relative to the existence of a strongly inaccessible cardinal. Zhou has also shown that $MA + \neg CH$ implies the existence of a Lindelöf first countable non-metrizable space with a small diagonal. It is not known, however, if it is consistent to have a (countably) compact first countable non-metrizable space with a small diagonal. In this section, we prove that (relative to a weakly compact cardinal) it is consistent with the negation of CH that each compact first countable space with a small diagonal is metrizable. Although Fleissner's condition above holds in the resulting model (35 of [DJW]) we do not know if the first countability assumption can be dropped.

To fit this problem into the framework of our method we first show that if $c = \aleph_2$ then it follows from $\mathcal{R}(\Gamma,\aleph_2,\varphi)$ for certain Γ and φ. Indeed, let Γ simply be the class of compact first countable spaces and let $\varphi(X,Z,A)$ be the formula "if $A = \{a_\alpha : \alpha \in \mu\} \subset X^2 - \Delta$, where μ is well ordered by $\in$ and has no countable cofinal set, then A does not τ-converge to Δ".

Proposition 5.1. If $c = \aleph_2$ and $\mathcal{R}(\Gamma,\aleph_2,\varphi)$ holds (with Γ and φ as above) then each compact first countable space with a small diagonal is metrizable.

Furthermore "first countable" can be dropped if $2^{\aleph_2} < \aleph_\omega$ and $\mathfrak{R}(\text{compact}, \aleph_2, \varphi)$ holds.

Proof. Let Y be a compact non-metrizable space. Let X be a compact non-metrizable subspace of Y such that X has a dense subset of cardinality at most $\aleph_2$ (recall that $\mathfrak{R}(\text{compact}, c^+, \text{metrizable})$ holds). Let μ be the minimum cardinality of a family of open subsets of X^2 such that $\Delta = \Delta_X$ is the intersection. Since X is compact and not metrizable $\omega < \mu$ and $\mu < \aleph_\omega$ (hence regular) since X has a dense set of cardinality $\aleph_2$. Since X is compact, μ is also the minimum cardinality of a neighbourhood base for Δ. Let $\{U_\alpha : \alpha < \mu\}$ be such a neighbourhood base and choose, inductively $a_\alpha \in \bigcap\{U_\beta : \beta < \alpha\} - [\Delta \cup \{a_\beta : \beta < \alpha\}] \subset X^2 - \Delta$ for $\alpha < \mu$. Now it is clear that $\varphi(X,\tau,A)$ fails where $A = \{a_\alpha : \alpha \subset \mu\}$. By assumption, there is a $Z \in [X]^{\aleph_1}$ such that $\varphi(Z, \tau|Z, \{a_\alpha : \alpha \in \mu\} \cap Z)$ fails. Therefore, $I = \{\alpha \in \mu : a_\alpha \in Z\}$ has no countable cofinal set and $\{a_\alpha : \alpha \in I\}$ converges to Δ_Z. Let J be a cofinal subset of I of order type ω_1 (I has cardinality $\aleph_1$ since $\{a_\alpha : \alpha < \mu\}$ is a $1-1$ indexing) and observe that each neighbourhood of Δ_Z (hence of Δ_X) contains all but a countable subset of $\{a_\alpha : \alpha \in J\}$. Therefore, Y does not have a small diagonal.

Lemma 5.2. If θ is weakly compact and G is $\mathrm{Mi}(\theta)$-generic then, in $V[G]$, $\mathfrak{R}(\Gamma,\aleph_2,\varphi)$ holds where Γ is "compact first countable" and $\varphi(X,\tau,A)$ is the formula "if $A = \{a_\alpha : \alpha \in I\} \subset X^2 - \Delta$ where I is well-ordered by $\in$ and has no countable cofinal set, then A does not τ-converge to Δ".

Proof. Let G be $\mathrm{Mi}(\theta)$-generic and suppose that $\langle X,\tau\rangle$ is in Γ and that $\varphi(X,\tau,A)$ does not hold. We may assume therefore, that $X = \theta$ and $\tau \subset \theta \times \omega \times \theta$ codes, as usual, the compact first countable topology on θ (we can assume all compact first countable topologies are on θ by Arkangelskii's result). We may also assume that μ is a regular cardinal with uncountable cofinality and that $A = \{a_\alpha : \alpha < \mu\} \subset \theta^2 \setminus \Delta$ converges to Δ in the τ-topology. Obviously, for each $\alpha < \mu$, $\{\beta < \mu : a_\alpha = a_\beta\}$ is not cofinal in μ hence we may assume that $a_\alpha \neq a_\beta$ for $\alpha < \beta < \mu \leq \theta$. It is not difficult to find Π^1_1 formulas which are equivalent to Γ and $\neg\varphi$ respectively. Let $\dot{\tau}$ and $\{\dot{a}_\alpha : \alpha < \mu\}$ be $\mathrm{Mi}(\theta)$-names for τ and $\{a_\alpha : \alpha < \mu\}$ respectively and assume that 1 forces that $\dot{\tau}$ and $\{\dot{a}_\alpha : \alpha < \mu\}$ have all the relevant properties. Now $1 \Vdash_{Mi(\theta)} \langle\theta,\dot{\tau}\rangle$ is compact and $\neg\varphi(\theta, \dot{\tau}, \{\dot{a}_\alpha : \alpha < \check{\mu}\})$ is a Π^1_1 sentence which holds in V_θ (see [DTW] for more about why this is Π^1_1 etc.). Since θ is weakly compact, there is a strongly inaccessible $\lambda < \theta$ such that V is a model of $1 \Vdash_{Mi(\lambda)} \langle\lambda, \dot{\tau} \cap V_\lambda\rangle$ is compact and $\neg\varphi(\lambda, \dot{\tau} \cap V_\lambda, \{\dot{a}_\alpha : \alpha < \mu\} \cap V_\lambda)$. Since all quantifiers in the above sentence can be restricted to subsets of V_λ we have that $1 \Vdash_{Mi(\lambda)} \langle\lambda, \dot{\tau} \cap V_\lambda\rangle$ is compact and $\neg\varphi(\lambda, \dot{\tau} \cap V_\lambda, \{\dot{a}_\alpha : \alpha < \mu\} \cap V_\lambda)$ also holds in V. Let G' be $G \cap Mi(\lambda)$ and let $\tau' = val(\dot{\tau} \cap V_\lambda, G')$. Let $I = \{\alpha \in \mu : \dot{a}_\alpha \in V_\lambda\}$ and $a'_\alpha = val(\dot{a}_\alpha, G')$ for $\alpha \in I$. Note that $\neg\varphi(\lambda, \tau', \{val(\dot{a}_\alpha, G') : \alpha \in I\})$ holds in $V[G']$. By 2.4, I does not have a countable cofinal set in $V[G]$, hence it remains only to prove that forcing with $Mi(\theta)$ $(\cong Mi(\theta)/Mi(\lambda))$ preserves that $\{a'_\alpha : \alpha \in I\}$ converges to Δ_λ (i.e. only Step 3 in the method remains). By 3.3, $\langle\lambda,\tau'\rangle$, hence Δ_λ, remains Lindelöf in $V[G]$. Therefore, if $U \subset \lambda^2$ is a neighbourhood of Δ_λ in $V[G]$, there is a countable subset $\mathcal{B}$ of the topology generated by τ' such that $\Delta_\lambda \subset \cup\{B \times B : B \in \mathcal{B}\} \subset U$. By 2.2 (iii), any countable subset of $V[G']$ in $V[G]$ is actually added by the Cohen real part of the forcing. Therefore, $\mathcal{B} \in V[\tilde{G}]$ where $\tilde{G}$ is G' union G restricted to the Cohen real part of $Mi(\theta)$ (i.e. the conditions whose support contains no odd ordinals above λ). By 3.4, there is a $\gamma \in I$ so that, in $V[\tilde{G}]$, $\{a'_\alpha : \alpha \in I - \gamma\} \subset \cup\{B \times B : B \in \mathcal{B}\}$. It follows that $\{a'_\alpha : \alpha \in I - \gamma\} \subset U$ (in $V[G]$), hence $\{a'_\alpha : \alpha \in I\}$ converges to Δ_λ.

Finally, our main theorem is a corollary to 5.1 and 5.2.

Theorem 5.3. If G is $Mi(\theta)$-generic, for a weakly compact θ, then, in $V[G]$, each compact first countable space with a small diagonal is metrizable.

Remark 5.4. It is frustrating that we can only prove 5.2 and 5.3 for first countable spaces. The place where this assumption is used in the proof is to show that Step 4 holds. As mentioned in the introduction there are weaker assumptions listed in [DTW] which still ensure that Step 4 is valid. In this case we could weaken the assumptions to "the space has countable tightness and countable subspaces are first countable". However, we do not even see how to weaken it down to Husek's original countable tightness assuption. Another possible strengthening of 5.3 is to assume that θ is only Mahlo or perhaps even strongly inaccessible. The place where we used the full strength of weak compactness was to reflect that "X is compact", i.e. to ensure that

$1 \Vdash_{Mi(\lambda)} \langle \lambda , \dot{\tau} \cap V_\lambda \rangle$ is compact. As we showed in 4.3, we can find a strongly inaccessible $\lambda < \theta$, if θ is Mahlo, so that $1 \Vdash_{Mi(\lambda)} \langle \lambda , \dot{\tau} \cap V_\lambda \rangle$ is $\aleph_1$-compact (and $\aleph_0$-compact). A space is called initially $\aleph_1$-compact if it is both $\aleph_0$-compact and $\aleph_1$-compact. The author and, independently, van Douwen have shown that CH implies that each initially $\aleph_1$-compact space with countable tightness is compact. It is an open question as to whether or not this holds in ZFC, however, it is very promising that Fremlin and Nyikos have recently shown that it follows from PFA.

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