MANY WEAK P-SETS

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To the memory of Kenneth Kunen

Abstract. It is known that for each continuous image of $\mathbb{N}^*$, there is a nowhere dense weak P-set of $\mathbb{N}^*$ that maps irreducibly onto it. We generalize this for every compact space of weight at most $\mathfrak{c}$. This allows us to show that there is a weak P-set in $\mathbb{N}^*$ which is homeomorphic to $\mathbb{N}^*$. This generalizes a result of the first-named author and answers a problem posed before 1990.

1. Introduction

Let $\mathbb{N}$ denote the discrete space of natural numbers. It is well-known that every countable subspace of $\mathbb{N}^*$ (=$\beta\mathbb{N} \setminus \mathbb{N}$) is C^*-embedded in $\beta\mathbb{N}$, [21, 6O(6)] (cf. [33, 1.5.2]). Hence if D is any countably infinite discrete subspace of $\beta\mathbb{N}$, then its closure $\overline{D}$ is homeomorphic to $\beta\mathbb{N}$. The remainder $\overline{D} \setminus D$ is contained in $\mathbb{N}^*$ and is what van Douwen called a *trivial* copy of $\mathbb{N}^*$ in $\mathbb{N}^*$. He asked around 1980[1] whether there also exist nontrivial copies of $\mathbb{N}^*$ in $\mathbb{N}^*$. This problem was not explicitly stated in one of his published papers. But it can be found as Question 20 in [23].

Non-trivial copies exist in abundance under the Continuum Hypothesis (abbreviated CH) by Parovičenko [34] (for example, the boundary of any noncompact open F_σ-subset of $\mathbb{N}^*$). They can be forced to exist in models of Martin's Axiom, and it was long suspected, motivated by results in [17, 3.14.2] and related earlier results in [25] that under the proper forcing axiom, PFA, these would not exist. In fact, Just [25] showed that it was consistent, and by [17] the same holds under PFA, that no nowhere dense closed P-set of $\mathbb{N}^*$ is homeomorphic to $\mathbb{N}^*$. This result is highly relevant to our present paper and we will come back to it below.

In [13], the first author answered van Douwen's problem in the affirmative: there *does* exist a nontrivial copy of $\mathbb{N}^*$ in $\mathbb{N}^*$. The proof was anticipated in [12], and used several tools that were developed in the theory of Čech-Stone compactifications following the results of Chae and Smith [5] and van Douwen [6] on remote points, and Kunen [28] on weak P-points. He used Aronszajn trees and remote point techniques to embed $\mathbb{N}^*$ in a nontrivial way in the absolute $E(2^{\omega_1})$ of 2^{ω_1}, and then applied the result in [31, 2.4] that $E(2^{\omega_1})$ can be embedded in $\mathbb{N}^*$ as a weak P-set to conclude the proof.

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[1]This is our best recollection.

The aim of this paper is to show that there is a nowhere dense weak P-set copy of $\mathbb{N}^*$ in $\mathbb{N}^*$. This is a significant improvement of [13] and answers a question that was around before 1990 (it was explicitly stated after Question 20 in [23] (we could not reconstruct its origin)). Hence Just's result just quoted cannot be generalized to weak P-sets. However, our nontrivial copy of $\mathbb{N}^*$ is not a $\mathfrak{c}$-OK set, so there is still room for improvement.

We say that a space X *maps irreducibly* onto a space Y, if there is a perfect map $f : X \to Y$ which is *irreducible*; this means that if A is a proper closed subset of X, then $f(A)$ is a proper closed subset of Y.

As we stated above, the proof in [13] used that for every $\mathbb{N}^*$ image there is a $\mathfrak{c}$-OK set in $\mathbb{N}^*$ that maps irreducibly onto it, [31, 2.4]. The condition about being an $\mathbb{N}^*$-image is a technical one needed in the proof. Our first main result is to show, somewhat unexpectedly, that the condition is superfluous.

Theorem 1.1. *For every compact space of weight at most* $\mathfrak{c}$*, there is a* $\mathfrak{c}$*-OK set in* $\mathbb{N}^*$ *that maps irreducibly onto it.*[2]

This new result allows for broader applications than the old one. To put this into perspective, we note the following. It is known from [34] that every compact space of weight at most $\aleph_1$ is an $\mathbb{N}^*$ image. Hence under CH, every compact space of weight $\mathfrak{c}$ is an $\mathbb{N}^*$ image. While the statement '*every compact space of weight* $\mathfrak{c}$ *is an* $\mathbb{N}^*$ *image*' is consistent with the failure of CH (see [4]), it *is* known to be consistent that there are many compact spaces of weight $\mathfrak{c}$ that are not $\mathbb{N}^*$ images. For those compact spaces there exist $\mathfrak{c}$-OK irreducible preimages by Theorem 1.1 but not by [31, 2.4]. An example of such a space is $\omega_2 + 1$ (Kunen [26, 12.7 and 12.3]). For more examples, see [33]. Other examples include the Stone space of the measure algebra of the real line, [14], and $\mathbb{N}^* \times \mathbb{N}^*$, [24]. For the proof of our main result, we need the Stone space of the measure algebra of 2^{ω_1}, which has weight $\mathfrak{c}$ and by the result in [14] is yet another example of a space for which it is consistent that it is not an $\mathbb{N}^*$ image.

As in [13], we use an Aronszajn tree in 2^{ω_1} in the proof of our main result. We let every node in the tree correspond to a remote point in the Stone space of a certain subalgebra of the measure algebra $\mathscr{M}_{\omega_1}$ on 2^{ω_1}. This allows us to conclude that the embedding we are after is indeed a weak P-set.

Theorem 1.2. *There is a nowhere dense weak P-set in* $\mathbb{N}^*$ *that is homeomorphic to* $\mathbb{N}^*$*.*

Since $\mathrm{st}(\mathscr{M}_{\omega_1})$ satisfies the countable chain condition, the nontrivial copy of $\mathbb{N}^*$ in $\mathbb{N}^*$ that we get from this result is not a $\mathfrak{c}$-OK set. This prompts the following problem.

Question 1.1. Is there a nowhere dense $\mathfrak{c}$-OK set in $\mathbb{N}^*$ that is homeomorphic to $\mathbb{N}^*$?

We will now briefly explain the history of Kunen's fundamental method from [28] for creating weak P-points in $\mathbb{N}^*$.

Its motivation came from questions about the homogeneity of Čech-Stone remainders. Rudin [35] showed that $\mathbb{N}^*$ contains a P-point under CH. From this he concluded that $\mathbb{N}^*$

[2] During final revision of the present paper, we came across Simon's paper [37] which contains a similar result.

is not homogeneous under CH. The same conclusion was reached by Frolík [19] in ZFC alone. His proof is based on a cardinality argument and does not yield two points in $\mathbb{N}^*$ with obvious different topological behavior. Whether or not there are P-points in $\mathbb{N}^*$ in ZFC remained a formidable open problem in topology and set theory for a long time, until it was settled by Shelah [39] in 1978: it is relatively consistent that P-points in $\mathbb{N}^*$ do *not* exist. Hence Rudin's method can not be used to give an 'honest proof' (this terminology is due to van Douwen) of the nonhomogeneity of $\mathbb{N}^*$. But help came from Kunen. A little earlier, he had a created in [27] a method for constructing special ultrafilters in ZFC. He used large independent families of sets in order to prevent certain transfinite constructions to stop prematurely. The ideas in [27] were refined in [28] for the construction of so-called $\mathfrak{c}$-OK points in $\mathbb{N}^*$ (in ZFC). Since every $\mathfrak{c}$-OK point is a weak P-point, this finally gave an 'honest proof' of the nonhomogeneity of $\mathbb{N}^*$. Instead of independent families, he used a combinatorially very complicated independent matrix of sets to achieve this result. His method for constructing special ultrafilters in ZFC was used extensively by himself (see e.g., [1,2]) and others (see e.g., [36], [8,10,13], [30–32], [38]) to get more general results in the same spirit. Kunen's method is also a central tool in the present paper.

2. Preliminaries

We follow Kunen [29, p. 11] to let '$\subset$' denote inclusion. Moreover, we let fin denote the ideal of finite subsets of $\mathbb{N}$. If $A \subset \mathbb{N}$, then A^* denotes the intersection of $\mathbb{N}^*$ and the closure of A in $\beta\mathbb{N}$.

If X is a space and $A \subset X$ is closed, then A is a *weak P-set* in X if for every countable subset D of $X \setminus A$, the closure of D and A are disjoint. A weak P-set consisting of a single point, is called a *weak P-point*.

A closed subset A of a space X is *κ-OK*, where κ is a cardinal, Kunen [28], if for every sequence of neighborhoods $\{U_n : n \in \omega\}$ of A, there is a κ-sequence of neighborhoods $\{V_\alpha : \alpha \in \kappa\}$ of A such that for all $n \geq 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \kappa$, $V_{\alpha_1} \cap \cdots \cap V_{\alpha_n} \subset U_n$. A κ-OK set consisting of a single point, is called a *κ-OK point*. It is clear that the property of κ-OK gets stronger if κ gets bigger. Moreover, every ω_1-OK set is a weak P-set, Kunen [28].

For a space X, we let $\mathrm{CO}(X)$ denote its Boolean algebra of clopen sets.

Let $\mathbb{B}$ be a Boolean algebra. The underlying space of its *Stone space*, $\mathrm{st}(\mathbb{B})$, is the set of all ultrafilters in $\mathbb{B}$. Its topology is generated by the collection $\{b^+ : b \in \mathbb{B}\}$, where $b^+ = \{u \in \mathrm{st}(\mathbb{B}) : b \in u\}$. It is well-known that every compact zero-dimensional space X is uniquely determined by $\mathrm{CO}(X)$.

A space X is *extremally disconnected* if the closure of any of its open subsets is open. It is well-known, and easy to prove, that a compact zero-dimensional space is extremally disconnected iff $\mathrm{CO}(X)$ is complete.

If $\mathbb{E}$ is a subalgebra of $\mathbb{B}$, then there is a natural continuous surjection $f^{\mathbb{B}}_{\mathbb{E}} : \mathrm{st}(\mathbb{B}) \to \mathrm{st}(\mathbb{E})$ defined by $f^{\mathbb{B}}_{\mathbb{E}}(u) = u \cap \mathbb{E}$. And if X and Y are zero-dimensional compact spaces for which there is a continuous surjection $f : X \to Y$, then $\mathbb{E} = \{f^{-1}(B) : B \in \mathrm{CO}(Y)\}$ is a subalgebra of $\mathrm{CO}(X)$. Moreover, $\mathrm{st}(\mathbb{E})$ and Y can be identified, and $f^{\mathbb{B}}_{\mathbb{E}}$ agrees with f

under this identification. This is of course nothing but the Stone Representation Theorem, the details of which can be found in [22].

Let $\mathbb{B}$ be a Boolean algebra, $\mathbb{E}$ a subalgebra of it for which there exists an element $b \in \mathbb{B} \setminus \mathbb{E}$ such that $\mathbb{B}$ is generated by $\mathbb{E} \cup \{b\}$. Let $X = \mathrm{st}(\mathbb{B})$, $Y = \mathrm{st}(\mathbb{E})$, and $f : X \to Y$ the canonical map. The element $b \in \mathbb{B}$ corresponds to a clopen subset B of X. Let C denote its complement. Put $B_Y = f(B)$ and $C_Y = f(C)$, respectively. Since $\mathbb{B}$ is generated by $\mathbb{E} \cup \{b\}$, is not difficult to show that $f \restriction B : B \to B_Y$ and $f \restriction C : C \to C_Y$ are homeomorphisms. This means that we can (and will) think of X as being homeomorphic to the subspace $X' = (B_Y \times \{0\}) \cup (C_Y \times \{1\})$ of $Y \times 2$. Hence f simply corresponds to the restriction of the projection $Y \times 2 \to Y$ to X'.

Let X be a compact zero-dimensional space of weight $\kappa \geq \omega$. Then the Boolean algebra $\mathrm{CO}(X)$ of clopen subsets of X has size κ. A moment's reflection shows that we can find subalgebras $\mathscr{B}_\alpha$ of $\mathrm{CO}(X)$ for $\alpha \leq \kappa$, such that

(1) $\mathscr{B}_0 = \{\emptyset, X\}$,
(2) $|\mathscr{B}_\alpha| \leq |\alpha| \cdot \omega$,
(3) if $\beta < \alpha$, then $\mathscr{B}_\beta \subset \mathscr{B}_\alpha$,
(4) $\mathscr{B}_\alpha = \bigcup_{\beta<\alpha} \mathscr{B}_\beta$, if α is a limit,
(5) if α is a successor, say $\alpha = \beta + 1$, then there is an element $B \in \mathscr{B}_\alpha \setminus \mathscr{B}_\beta$, such that $\mathscr{B}_\alpha$ is generated by $\{B\} \cup \mathscr{B}_\beta$,
(6) $\mathscr{B}_\kappa = \mathrm{CO}(X)$.

We now recall the notion of an independent linked family from Kunen [28], which is one of the central concepts in this paper.

Definition 2.1. *Let F be closed subset of $\mathbb{N}^*$, X a compact space and $f : F \to X$ a continuous surjection.*

(a) *If $1 \leq n \in \omega$, an indexed family $\{A_i : i \in I\}$ of infinite subsets of $\mathbb{N}$ is precisely n-linked with respect to (w.r.t) $\langle F, f\rangle$, iff for all $\sigma \in [I]^n$, $f(F \cap \bigcap_{i\in\sigma} A_i^*) = X$ but for all $\sigma \in [I]^{n+1}$, $\bigcap_{i\in\sigma} A_i \in \mathrm{fin}$.*
(b) *An indexed family $\{A_{in} : i \in I, 1 \leq n \in \omega\}$ is a linked system w.r.t. $\langle F, f\rangle$ iff for each n, $\{A_{in} : i \in I\}$ is precisely n-linked w.r.t. $\langle F, f\rangle$, and for each n and i, $A_{in} \subset A_{i,n+1}$,*
(c) *An indexed family $\{A^j_{in} : i \in I, 1 \leq n \in \omega, j \in J\}$ is an I by J independent linked family w.r.t. $\langle F, f\rangle$ iff for each $j \in J$, $\{A^j_{in} : i \in I, 1 \leq n \in \omega\}$ is a linked system w.r.t. $\langle F, f\rangle$, and:*
$$f\big(F \cap \bigcap_{j\in\tau} \bigcap_{i\in\sigma_j} (A^j_{in_j})^*\big) = X,$$
whenever $\tau \in [J]^{<\aleph_0}$, and for each $j \in \tau, 1 \leq n_j \in \omega$ and $\sigma_j \in [I]^{n_j}$.

Lemma 2.2 (Kunen [28, 2.2])**.** *There is a $\mathfrak{c}$ by $\mathfrak{c}$ independent family w.r.t. $\langle F, f\rangle$, where $F = \mathbb{N}^*$ and $f : F \to \{0\}$ is the constant function with values 0.*

If X is a space then βX denotes its Čech-Stone compactification. Moreover, X^* denotes $\beta X \setminus X$.

3. The proof of Theorem 1.1

Let X be a compact space of weight at most $\mathfrak{c}$.

For the proof of Theorem 1.1, it suffices to assume that the weight of X is $\mathfrak{c}$. Indeed, if X has weight less than $\mathfrak{c}$, then consider the topological sum $Y = (X \times \{0\}) \cup (2^{\mathfrak{c}} \times \{1\})$ of X and $2^{\mathfrak{c}}$. Then Y has weight $\mathfrak{c}$, and if Z is a $\mathfrak{c}$-OK set in $\mathbb{N}^*$ that admits an irreducible map π onto Y, then $\pi^{-1}(X \times \{0\})$ is a $\mathfrak{c}$-OK set in $\mathbb{N}^*$ as well and $\pi \restriction \pi^{-1}(X \times \{0\})$ is irreducible.

It also suffices to assume that X is zero-dimensional, [33, 1.3.2].

So, to begin with the actual proof, let X be an arbitrary compact zero-dimensional space of weight $\mathfrak{c}$. Moreover, write $\mathrm{CO}(X)$ as $\bigcup_{\alpha \le \mathfrak{c}} \mathscr{B}_\alpha$, where the $\mathscr{B}_\alpha$'s are as in §2. For every $\alpha \le \mathfrak{c}$, put $X_\alpha = \mathrm{st}(\mathscr{B}_\alpha)$. Observe that X_0 is a single point. Moreover, for $\beta < \alpha \le \mathfrak{c}$, let $f^\alpha_\beta : X_\alpha \to X_\beta$ be the canonical continuous surjection.

Let $\{G_\mu : (\mu < \mathfrak{c}) \,\&\, (\mu \text{ even})\}$ enumerate all nonempty clopen subsets of $\mathbb{N}^*$. Moreover, let $\{\langle C_{\mu n} : n \in \omega\rangle : (\mu < \mathfrak{c}) \,\&\, (\mu \text{ odd})\}$ enumerate all sequences of infinite subsets of $\mathbb{N}$ such that $C_{\mu,n+1} \subset C_{\mu n}$ for each $n \in \omega$. We assume that each sequence is listed cofinally often.

Let $\mathscr{A} = \{A^\beta_{\alpha n} : \alpha < \mathfrak{c}, 1 \le n \in \omega, \beta < \mathfrak{c}\}$ be a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family w.r.t. $\langle \mathbb{N}^*, f\rangle$ (Lemma 2.2).

By induction on $\mu < \mathfrak{c}$ we construct F_μ, $f_\mu : F_\mu \to X_\mu$ and K_μ such that

(1) F_μ is a closed subset of $\mathbb{N}^*$, $f_\mu : F_\mu \to X_\mu$ is a continuous surjection, $K_\mu \subset \mathfrak{c}$, and $\{A^\beta_{\alpha n} : \alpha < \mathfrak{c}, 1 \le n \in \omega, \beta \in K_\mu\}$ is an independent linked family w.r.t. $\langle F_\mu, f_\mu\rangle$;
(2) $K_0 = \mathfrak{c}$, $F_0 = \mathbb{N}^*$, and $f_0 = f$;
(3) $\nu < \mu$ implies $F_\mu \subset F_\nu$, $K_\mu \subset K_\nu$, and the diagram

$$\begin{array}{ccc} F_\nu & \hookleftarrow & F_\mu \\ {\scriptstyle f_\nu}\downarrow & & \downarrow{\scriptstyle f_\mu} \\ X_\nu & \xleftarrow{\;f^\mu_\nu\;} & X_\mu \end{array}$$

commutes;
(4) if μ is a limit ordinal, $F_\mu = \bigcap_{\nu<\mu} F_\nu$, and $K_\mu = \bigcap_{\nu<\mu} K_\nu$;
(5) for each μ, $K_\mu \setminus K_{\mu+1}$ is finite;
(6) if μ is even, either $F_{\mu+1} \subset G_\mu$, or $f_{\mu+1}(G_\mu \cap F_{\mu+1}) \ne X_{\mu+1}$;
(7) if μ is odd, and $F_\mu \subset C^*_{\mu n}$ for every n, then there are clopen neighborhoods $E_{\mu\alpha}$ of $F_{\mu+1}$ for $\alpha < \mathfrak{c}$ in $\mathbb{N}^*$ such that for all $n \ge 1$ and all $\alpha_1 < \alpha_2 < \cdots \alpha_n < \mathfrak{c}$ we have $E_{\mu\alpha_1} \cap \cdots \cap E_{\mu\alpha_n} \subset C^*_{\mu n}$.

Let us assume for a moment that this construction can indeed be carried out. Put $F_{\mathfrak{c}} = \bigcap_{\mu<\mathfrak{c}} F_\mu$. By compactness and the commutativity of the diagrams in (3), there is a continuous surjection $f_{\mathfrak{c}} : F_{\mathfrak{c}} \to X_{\mathfrak{c}} = X$ such that for every $\mu \le \mathfrak{c}$, the diagram

$$\begin{array}{ccc} F_\mu & \hookleftarrow & F_{\mathfrak{c}} \\ {\scriptstyle f_\mu}\downarrow & & \downarrow{\scriptstyle f_{\mathfrak{c}}} \\ X_\mu & \xleftarrow{f^{\mathfrak{c}}_\mu} & X_{\mathfrak{c}} \end{array}$$

commutes. Observe that by (7), $F_{\mathfrak{c}}$ is a $\mathfrak{c}$-OK-set in $\mathbb{N}^*$. Hence it suffices to check that $f_{\mathfrak{c}}$ is irreducible. To check this, let B be a proper closed subset of F. For some $\mu < \mathfrak{c}$, $B \subset G_\mu$ and $F \setminus G_\mu \neq \emptyset$. Hence $F_{\mu+1} \not\subset G_\mu$, and so by (6), $f_{\mu+1}(G_\mu \cap F_{\mu+1}) \neq X_{\mu+1}$. Hence, clearly, $f_{\mathfrak{c}}(B \cap F_{\mathfrak{c}}) \neq X_{\mathfrak{c}}$.

Fix $\mu < \mathfrak{c}$, and assume that the F_ν, f_ν and K_μ have been found, for each $\nu < \mu$. If μ is a limit, then F_μ and K_μ are determined by (4). Moreover, $f_\mu : F_\mu \to X_\mu$ is uniquely determined by the commutativity of the diagrams in (3), and (1) is immediate from compactness. Hence we may assume in fact that the F_ν, f_ν and K_ν have been found, for each $\nu \le \mu$. We will construct $F_{\mu+1}$, $f_{\mu+1} : F_{\mu+1} \to X_{\mu+1}$ and $K_{\mu+1}$.

Take an arbitrary $\beta_0 \in K_\mu$, and consider the sets $C_0 = A^{\beta_0}_{1,0}$ and $C_1 = \mathbb{N} \setminus C_0$ and observe that $A^{\beta_0}_{1,1}$ is almost contained in C_1. Hence for $k < 2$, $f_\mu(F_\mu \cap C_k^*) = X_\mu$ by our inductive assumptions.

Consider the space $X_{\mu+1}$. As we saw above, there is a closed cover $\{S_0, S_1\}$ of X_μ such that $X_{\mu+1}$ can be thought of as the the subspace $X'_{\mu+1} = (S_0 \times \{0\}) \cup (S_1 \times \{1\})$ of $X_\mu \times 2$, and $f^{\mu+1}_\mu : X_{\mu+1} \to X_\mu$ is the restriction of the projection $X_\mu \times 2 \to X_\mu$ to $X_{\mu+1}$.

For $k < 2$, put $L_k = f_\mu^{-1}(S_k) \cap (F_\mu \cap C_k^*)$. Clearly, $f_\mu(L_k) = S_k$ for $k < 2$. Define $g : Y = L_0 \cup L_1 \to X_{\mu+1}$ by $g(x) = \langle f_\mu(x), k\rangle$ if $x \in L_k$, $k < 2$. Observe that $f^{\mu+1}_\mu \circ g = f_\mu$.

Claim 1. $\{A^\beta_{\alpha n} : \alpha < \mathfrak{c}, 1 \le n \in \omega, \beta \in K_\mu \setminus \{\beta_0\}\}$ is an independent linked family w.r.t. $\langle Y, g\rangle$.

Indeed, let $\tau \in [K_\mu \setminus \{\beta_0\}]^{<\aleph_0}$, and for $j \in \tau, 1 \le n_j \in \omega$ and $\sigma_j \in [\mathfrak{c}]^{n_j}$. Then by our inductive assumptions, for $k < 2$ and $P = \bigcap_{j\in\tau} \bigcap_{i\in\sigma_j} (A^j_{in_j})^*$ we have

$$f_\mu(L_k \cap P) = f_\mu(f_\mu^{-1}(S_k) \cap (F_\mu \cap C_k^*) \cap P) = S_k \cap f_\mu(F_\mu \cap C_k^* \cap P) = S_k,$$

from which the Claim is obvious.

Case 1. μ is even.

Put $T = G_\mu \cap Y$. If $\{A^\beta_{\alpha n} : \alpha < \mathfrak{c}, 1 \le n \in \omega, \beta \in K_\mu \setminus \{\beta_0\}\}$ is an independent linked family w.r.t. $\langle T, g \restriction T\rangle$, we set $F_{\mu+1} = T$, $K_{\mu+1} = K_\mu \setminus \{\beta_0\}$ and $f_{\mu+1} = g \restriction T$. If not, there is $\tau \in [K_\mu \setminus \{\beta_0\}]^{<\aleph_0}$, and for each $j \in \tau$, $1 \le n_j \in \omega$ and $\sigma_j \in [\mathfrak{c}]^{n_j}$ such that

$$g(Y \cap \bigcap_{j\in\tau} \bigcap_{i\in\sigma_j} (A^j_{in_j})^*) \neq X_{\mu+1}.$$

Then let $F_{\mu+1} = Y \cap \bigcap_{j\in\tau} \bigcap_{i\in\sigma_j} (A^j_{in_j})^*$, $K_{\mu+1} = K_\mu \setminus (\{\beta_0\} \cup \tau)$, and $f_{\mu+1} = g \restriction F_{\mu+1}$.

Case 2. μ is odd.

Assume first that there exists n such that $Y \not\subset C^*_{\mu n}$. Then put $F_{\mu+1} = Y$, $K_{\mu+1} = K_\mu \setminus \{\beta_0\}$ and $f_{\mu+1} = g \restriction Y$. If not, fix $\beta \in K_\mu \setminus \{\beta_0\}$, and let $K_{\mu+1} = K_\mu \setminus \{\beta_0, \beta\}$. For every $\alpha < \mathfrak{c}$, put $D_{\mu\alpha} = \bigcup_{1 \leq n \in \omega} A^\beta_{\alpha n} \cap C_{\mu n}$, $F_{\mu+1} = Y \cap \bigcap_{\alpha<\mathfrak{c}} D^*_{\mu\alpha}$, and $f_{\mu+1} = g \restriction F_{\mu+1}$. We will show that these choices satisfy our inductive requirements.

That $\{A^\beta_{\alpha n} : \alpha < \mathfrak{c}, 1 \leq n \in \omega, \beta \in K_{\mu+1}\}$ is an independent linked family w.r.t. $\langle F_{\mu+1}, f_{\mu+1}\rangle$ is easy. By Claim 1, it suffices to observe that for arbitrary $\alpha_1 < \cdots < \alpha_n < \mathfrak{c}$,

$$\bigcap_{k=1}^n D^*_{\mu\alpha_k} \cap Y \supseteq \bigcap_{k=1}^n (A^\beta_{\alpha_k n})^* \cap C^*_{\mu n} \cap Y = \bigcap_{k=1}^n (A^\beta_{\alpha_k n})^* \cap Y.$$

To verify condition (7), let $\alpha_1 < \cdots < \alpha_n < \mathfrak{c}$, and put

$$S = (D_{\mu\alpha_1} \cap \cdots \cap D_{\mu\alpha_n}) \setminus C_{\mu n}.$$

If $n = 1$, then $S = \emptyset$. If $n > 1$, then

$$S \subset A^\beta_{\alpha_1, n-1} \cap \cdots \cap A^\beta_{\alpha_n, n-1}.$$

Hence S is finite since these $A^\beta_{\alpha_j, n-1}$ are precisely $(n-1)$-linked. From this we conclude that

$$(D^*_{\mu\alpha_1} \cap \cdots \cap D^*_{\mu\alpha_n}) \subset C^*_{\mu n},$$

as required.

The above proof was inspired by the proof of Theorem 3.1 in Kunen [28].

With precisely the same proof, we can generalize [31, 2.4], as follows. (We will not need this generalization in the rest of the proof. For the definition of nice filter, see [31].)

Theorem 3.1. *Let $X = \omega \times Z$, where Z is a nonempty compact space of weight at most $\mathfrak{c}$ and suppose that $\mathscr{F}$ is a nice filter on X. For any compact space Y of weight at most $\mathfrak{c}$, there is a $\mathfrak{c}$-OK set A in X^* such that $A \subset \bigcap_{F \in \mathscr{F}} F^*$ that maps irreducibly onto Y.*

Corollary 3.2. *Let $X = \omega \times Z_n$, where each Z_n is a nonempty compact space of weight at most $\mathfrak{c}$ and suppose that $\mathscr{F}$ is a nice filter on X. For any compact space Y of weight at most $\mathfrak{c}$, there is a $\mathfrak{c}$-OK set A in X^* such that $A \subset \bigcap_{F \in \mathscr{F}} F^*$ that maps irreducibly onto Y.*

Proof. Let Z be the one-point compactification of the topological sum of the spaces Z_n. The nice filter $\mathscr{F}$ on X is also a nice filter on $\omega \times Z$. Hence the more general result is an immediate consequence of the previous result. □

Remark 1. By taking for $\mathscr{F}$ the co-finite filter on ω, we see that Theorem 1.1 is a consequence of Corollary 5.3.

4. Proof of Theorem 1.2

4.1. **Measure algebras.** For all undefined notions on measure theory, see Fremlin [18].

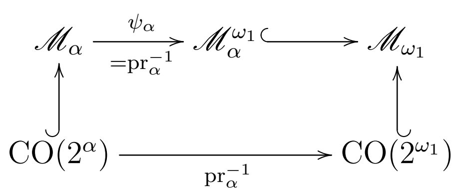
We let μ denote the standard Haar measure on the elements of the σ-algebra generated by the basic clopen sets, $\mathrm{CO}(2^{\omega_1})$, of 2^{ω_1}. Hence for $s \in \mathrm{Fn}(\omega_1, 2)$, $\mu([s])$ is equal to $2^{-|\operatorname{dom}(s)|}$, where $[s]$ denotes the usual basic clopen subset of 2^{ω_1} determined by s.

Let $\mathscr{M}_{\omega_1}$ denote the measure algebra on 2^{ω_1}. That is, $\mathscr{M}_{\omega_1}$ consists of equivalence classes of measurable subsets of 2^{ω_1}, where two measurable sets are *equivalent* when their symmetric difference has measure 0.

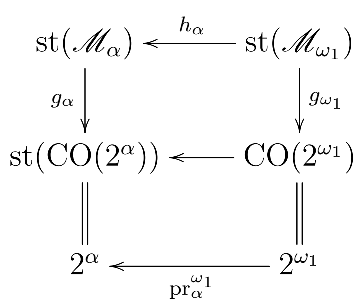
We treat $\mathrm{CO}(2^{\omega_1})$ as a subalgebra of $\mathscr{M}_{\omega_1}$. That is, we identify each $[s]$ where $s \in \mathrm{Fn}(\omega_1, 2)$ with its own equivalence class in $\mathscr{M}_{\omega_1}$.

In case $\alpha \in \omega_1$ and $\mathrm{dom}(s) \subset \alpha$, we let $[s]_\alpha$ be the associated clopen subset of 2^α. Let pr_α denote the projection mapping from 2^{ω_1} onto 2^α where, $\mathrm{pr}_\alpha(x) = x \restriction \alpha$ for $x \in 2^{\omega_1}$. If needed, we will use pr^β_α for the projection map from 2^β onto 2^α in case $\alpha \le \beta$. Of course $[s]_\alpha = \mathrm{pr}_\alpha([s])$.

For $\omega \le \alpha \in \omega_1$ we let $\mathscr{M}_\alpha$ denote the measure algebra on 2^α. Again, we treat $\mathrm{CO}(2^\alpha)$ as a subalgebra of $\mathscr{M}_\alpha$. We will abuse notation and use $\mu(b)$ to also denote the measure of $b \in \mathscr{M}_\alpha$ since it should cause no confusion. The mapping $\psi_\alpha = \mathrm{pr}_\alpha^{-1}$ denotes a canonical (measure preserving) embedding of $\mathscr{M}_\alpha$ into $\mathscr{M}_{\omega_1}$. We will let $\mathscr{M}^{\omega_1}_\alpha$ denote the range of ψ_α. This leads to the following diagram:



We note that $\mathscr{M}_{\omega_1}$ is equal to the union of the family $\{\mathscr{M}^{\omega_1}_\alpha : \alpha \in \omega_1\}$. For $\alpha \le \omega_1$, let g_α be the induced canonical mapping from $\mathrm{st}(\mathscr{M}_\alpha)$ onto $\mathrm{st}(\mathrm{CO}(2^\alpha))(= 2^\alpha)$. Similarly, there is a canonical map from $\mathrm{st}(\mathscr{M}_{\omega_1})$ onto $\mathrm{st}(\mathscr{M}_\alpha)$ which we will denote by h_α. This leads to the following diagram:



The definition of $h_\alpha(\mathscr{U})$ for $\mathscr{U} \in \mathrm{st}(\mathscr{M}_{\omega_1})$ is the composition of $\mathscr{U} \mapsto (\mathscr{U} \cap \mathscr{M}^{\omega_1}_\alpha) \mapsto \psi_\alpha^{-1}(\mathscr{U} \cap \mathscr{M}^{\omega_1}_\alpha)$. And since we identify 2^α with $\mathrm{st}(\mathrm{CO}(2^\alpha))$, $g_\alpha(\mathscr{U})$ for $\mathscr{U} \in \mathrm{st}(\mathscr{M}_\alpha)$ is equal to $\mathscr{U} \cap \mathrm{CO}(2^\alpha)$.

4.2. **Aronszajn trees.** Let $T \subset 2^{<\omega_1}$ be an Aronszajn tree; specifically, T is downward closed, has no maximal elements, has no uncountable branches, and for $\alpha < \beta \in \omega_1$, $T_\alpha = T \cap 2^\alpha$ is countable and for each $t \in T_\alpha$, there is an extension of t in T_β.

We first note that we may assume that there is a sequence $\{t(\omega, n) : n \in \omega\} \subset T_\omega$ satisfying that $\{t_n = t(\omega, n) \restriction n{+}1 : n \in \omega\}$ is an antichain in $2^{<\omega}$. Hence the collection of basic clopen sets $\{[t_n] : n \in \omega\}$ in 2^{ω_1} is pairwise disjoint.

The following result is a consequence of [13, Lemma 2.9] (see also [11]).

Lemma 4.1. *There is a sequence* $\mathscr{T} = \{t(\alpha, n) : \omega \leq \alpha \in \omega_1, n \in \omega\} \subset T$ *satisfying, for all* $k \in \omega$ *and increasing* $\{\alpha_j : j \leq k\} \subset \omega_1 \setminus \omega$,

(1) $t(\omega, n) = t(\alpha_k, n) \restriction \omega$,
(2) *there is an* $\bar{n} \in \omega$ *such that, for all* $j \leq k$ *and* $n > \bar{n}$, $t(\alpha_j, n) = t(\alpha_k, n) \restriction \alpha_j$.

Proof. By [13, Lemma 2.9], there exists a family $\{s(\alpha, n) : \alpha \in \omega_1, n \in \omega\} \subset T$ such that

(3) for each $\alpha \in \omega_1$, $\{s(\alpha, n) : n \in \omega\} \subset T_\alpha$,
(4) for $\beta < \alpha$, there is an $n \in \omega$ so that for all $k \geq n$, $s(\beta, k) \subset s(\alpha, k)$.

The proof of [13, Lemma 2.9] shows that it is easy to arrange that $t(\omega, n) \subset s(\alpha, n)$ for all $\omega < \alpha$. (Here the $t(\omega, n)$'s are as above.) Hence it is clear that by putting $t(\alpha, n) = s(\alpha, n)$ for all $\alpha \in \omega_1 \setminus \omega$ and $n \in \omega$, we have what we are looking for. □

Since $t(\alpha, n)$ is a point in 2^α, we will abuse notation and let $\tilde{t}(\alpha, n)$ denote the G_δ-subset $g_\alpha^{-1}(\{t(\alpha, n)\})$ of $\mathrm{st}(\mathscr{M}_\alpha)$.

The following result, which also was used in [13, Lemma 2.12], follows easily from the fact that T does not have cofinal branches.

Lemma 4.2. *If* D *is any countable subset of* 2^{ω_1}, *there is a* $\delta \in \omega_1$ *such that* $\mathrm{pr}_\delta(D)$ *is disjoint from* T_δ. *Hence for all* $\delta \leq \alpha \in \omega_1$, $\mathrm{pr}_\alpha(D)$ *is disjoint from* $\{t(\alpha, n) : n \in \omega\}$.

Corollary 4.3. *If* D *is any countable subset of* $\mathrm{st}(\mathscr{M}_{\omega_1})$, *there is a* $\delta \in \omega_1$ *such that* $h_\delta(D)$ *is disjoint from* $\bigcup\{\tilde{t}(\delta, n) : n \in \omega\}$.

4.3. **Remote points.** The point $x \in X^*$ is called a *remote point* of X if $x \notin \overline{A}$ for each nowhere dense subset A of X. Here closure means closure in βX. It was shown by van Douwen [6] and, independently, Chae and Smith [5], that every nonpseudocompact space of countable π-weight has a remote point. Not all nonpseudocompact spaces have remote points, [7]. The most general result about the existence of remote points is [9], where it was shown that every nonpseudocompact space which satisfies the countable chain condition and has π-weight at most ω_1, has a remote point. Ideas that come from remote point proofs were used frequently in set theoretic topology in the last decades, and for various unrelated applications. For an example of this phenomenon in forcing, see e.g., [15, 16].

We will make good use of remote points in the proof of our main result.

Say that an ultrafilter (point) $\mathscr{U} \in \mathrm{st}(\mathscr{M}_\alpha)$ is $\tilde{t}(\alpha, n)$-remote if

(1) $g_\alpha(\mathscr{U}) = t(\alpha, n)$, and
(2) $\mathscr{U}$ is not in the closure of any nowhere dense subset of $\mathrm{st}(\mathscr{M}_\alpha) \setminus \tilde{t}(\alpha, n)$.

We will say that a filter $\mathscr{F}$ on $\mathscr{M}_\alpha$ is $\tilde{t}(\alpha, n)$-remote if $\mathscr{U}$ is $\tilde{t}(\alpha, n)$-remote for every $\mathscr{F} \subset \mathscr{U} \in \mathrm{st}(\mathscr{M}_\alpha)$. Of course one could also consider defining a notion like $\tilde{t}(\alpha, n)$-weak-P or countable remote or whatever.

Definition 4.4. *A* $\mathscr{T}$*-sequence of ultrafilters is a sequence*

$$\mathscr{U}_\mathscr{T} = \{\mathscr{U}(\alpha, n) : \omega \leq \alpha \in \omega_1,\ n \in \omega\}$$

satisfying, for each $\omega \leq \alpha \leq \beta \in \omega_1$:

(1) $\mathscr{U}(\alpha, n) \in \mathrm{st}(\mathscr{M}_\alpha)$ *and* $g_\alpha(\mathscr{U}(\alpha, n)) = t(\alpha, n)$ *for all* $n \in \omega$,

(2) *for all but finitely many* $n \in \omega$, $\psi_\alpha(\mathscr{U}(\alpha, n))$ *is a subset of* $\psi_\beta(\mathscr{U}(\beta, n))$.

A $\mathscr{T}$*-sequence of remote ultrafilters will mean that* $\mathscr{U}(\alpha, n)$ *is* $\tilde{t}(\alpha, n)$*-remote for all* $\omega \leq \alpha \in \omega_1$ *and* $n \in \omega$.

We let $K(\mathscr{U}_\mathscr{T})$ *denote the closed subset of* $\mathrm{st}(\mathscr{M}_{\omega_1})$ *that is equal to*

$$(\dagger) \qquad \bigcap\{b^+ : (\exists\, \omega \leq \alpha \in \omega_1)(\exists\, m \in \omega)(\forall\, n > m) \;\; b \in \psi_\alpha(\mathscr{U}(\alpha, n)\}.$$

Let us first discuss where the set $K(\mathscr{U}_\mathscr{T})$ is placed in $\mathrm{st}(\mathscr{M}_{\omega_1})$. Recall the antichain $\{t_n : n \in \omega\} \subset 2^{<\omega}$ described in the definition of the sequence $\mathscr{T}$. We have adopted the convention that we may regard $[t_n]$ as a member of $\mathscr{M}_{\omega_1}$ and thus $\{[t_n]^+ : n \in \omega\}$ is a sequence of pairwise disjoint nonempty clopen subsets of $\mathrm{st}(\mathscr{M}_{\omega_1})$. By condition (1) in Lemma 4.1, $[t_n]^+ \in \psi_\alpha(\mathscr{U}(\alpha, n))$ for every $\omega \leq \alpha \in \omega_1$ and $n \in \omega$. Let B be the closure of $\bigcup_{n\in\omega}[t_n]^+$ in $\mathrm{st}(\mathscr{M}_{\omega_1})$. By extremal disconnectivity of $\mathrm{st}(\mathscr{M}_{\omega_1})$, there exists $\underline{b} \in \mathscr{M}_{\omega_1}$ such that $\underline{b}^+ = B$. For every $n \in \omega$, let $b_n = \underline{b} \setminus \bigcup_{m\leq n}[t_m]^+$. Clearly, each b_n is one of the elements in $\mathscr{M}_{\omega_1}$ that satisfies the condition in the definition of $K(\mathscr{U}_\mathscr{T})$. This means that

$$K(\mathscr{U}_\mathscr{T}) \subset \underline{b}^+ \setminus \bigcup_{n\in\omega}[t_n]^+.$$

Let $g : \bigcup_{n\in\omega}[t_n]^+ \to \omega$ be such that $g([t_n]^+) = n$ for all $n \in \omega$. Since $\mathrm{st}(\mathscr{M}_{\omega_1})$ is extremally disconnected, the closure $\underline{b}^+$ of $\bigcup_{n\in\omega}[t_n]^+$ is its Čech-Stone compactification, [33, 1.2.2], hence g can be extended to a continuous map f from $\underline{b}^+$ to $\beta\omega$ satisfying that $f([t_n]^+) = n$ for all $n \in \omega$.

We are now ready for the proof of our next main result in which we exploit the remote points and make similar use of the properties of the Aronszajn tree from above, as in [13, 2.10].

Theorem 4.5. *If* $\mathscr{U}_\mathscr{T}$ *is a* $\mathscr{T}$*-sequence of remote ultrafilters, then* $K(\mathscr{U}_\mathscr{T})$ *is a weak P-set in* $\mathrm{st}(\mathscr{M}_{\omega_1})$ *and is homeomorphic to* $\mathbb{N}^*$.

Proof. We prove first that $K = K(\mathscr{U}_\mathscr{T})$ is a weak P-set. Let D be any countable subset of $\mathrm{st}(\mathscr{M}_{\omega_1})$ that is disjoint from K. Choose $\delta \in \omega_1$ large enough so that, for each $d \in D$ there is an $\alpha < \delta$ and a $b_d \in \mathscr{M}_\alpha^{\omega_1} \setminus d$ such that $b_d \in \psi_\alpha(\mathscr{U}(\alpha, n))$ for all but finitely many $n \in \omega$. This uses simply that $d \notin K(\mathscr{U}_\mathscr{T})$. By possibly increasing δ, we can also assume that $h_\delta(D)$ is disjoint from $\bigcup\{\tilde{t}(\delta, n) : n \in \omega\}$ as per Corollary 4.3. Since $U(\delta, n)$ for $n \in \omega$ is $\tilde{t}(\delta, n)$ remote, this implies that the closure in $\mathrm{st}(\mathscr{M}_\delta)$ of $h_\delta(D)$ is disjoint from the sequence $\{\mathscr{U}(\delta, n) : n \in \omega\}$. Fix any $d \in D$ and choose $n_d \in \omega$ so that $b_d \in \psi_\delta(\mathscr{U}(\delta, n))$ for all $n > n_d$ (using Definition 4.4 (2)). Since $b_d \in \mathscr{M}_\delta^{\omega_1}$ and $b_d \notin d$, we have that $\psi_\delta^{-1}(b_d) \notin h_\delta(d)$. Topologically, this means that $\mathrm{st}(\mathscr{M}_\delta) \setminus h_\delta(b_d^+)$ is a neighorhood of $h_\delta(d)$ that meets $\{\mathscr{U}(\delta, n) : n \in \omega\}$ in a finite set. This proves that $A = h_\delta(D)$ and $B = \{\mathscr{U}(\delta, n) : n \in \omega\}$ are weakly separated (i.e. $\overline{A} \cap B$ and $A \cap \overline{B}$ are empty). Since $\mathrm{st}(\mathscr{M}_\delta)$ is extremally disconnected, weakly separated countable sets have disjoint closures, [21, Problem 9H]. Therefore there is a $b \in \mathscr{M}_\delta$ satisfying that $b \in \mathscr{U}(\delta, n)$ for all $n \in \omega$ and $b \notin h_\delta(d)$ for all $d \in D$. Of course this means that $K \subset \psi_\delta(b)^+$ and $\psi_\delta(b)^+ \cap D = \emptyset$.

Now prove that K is homeomorphic to $\mathbb{N}^*$. Consider the element $b \in \mathscr{M}_{\omega_1}$ and the map $f : b^+ \to \beta\omega$, discussed right before the formulation of the theorem. Observe that $f(K) \subset \omega^*$. We will show that $f \restriction K$ is 1-to-1 and onto ω^*.

To prove that $f \restriction K$ is onto we consider any finitely many $\{b_j^+ : j \leq k\} \subset \mathscr{M}_{\omega_1}$ as in the definition of $K(\mathscr{U}_{\mathscr{T}})$ (see (†)). Choose, for each $j \leq k$, $\alpha_j \in \omega_1$ and m_j so that $b_j \in \psi_{\alpha_j}(\mathscr{U}(\alpha_j, n))$ for all $n > m_j$. We may assume that $\alpha_j \leq \alpha_k$ for all $j \leq k$. By condition (2) of Definition 4.4, there is an m such that, for all $n > m$, $b = \wedge\{b_j : j \leq k\}$ is an element of $\psi_{\alpha_k}(\mathscr{U}(\alpha_k, n))$. Of course $[t_n]_{\alpha_k} \in \mathscr{U}(\alpha_k, n)$ for all n, and this shows that $b^+ \cap [t_n] \neq \emptyset$ for all $n > m$. It follows that $f(b^+)$ contains the closure of $\omega \setminus m$. Hence, by what we observed above, $f(K) = \omega^*$.

Now we prove that $f \restriction K$ is 1-to-1. Let $\mathscr{U}$ and $\mathscr{W}$ be distinct elements of K. Choose any $b \in \mathscr{M}_{\omega_1}$ such that $b \in \mathscr{U} \setminus \mathscr{W}$. We may choose $\alpha \in \omega_1$ such that $b \in \mathscr{M}_\alpha^{\omega_1}$. Let $A = \{n : b \in \psi_\alpha(\mathscr{U}(\alpha, n))\}$ and let $a \in \mathscr{M}_{\omega_1}$ be the element representing the equivalence class of the Borel set $\bigcup\{[t_n] \setminus b : n \in A\} \cup \bigcup\{[t_n] : n \in \omega \setminus A\}$ in 2^{ω_1}. Since $(a \cup b)^+ = a^+ \cup b^+$ and, clearly $K \subset (a \cup b)^+$, it follows that $a \in \mathscr{W}$. Finally, we note that A is in the ultrafilter $f(\mathscr{U})$ and $\omega \setminus A$ is in $f(\mathscr{W})$. □

4.4. **Existence of sequences of remote ultrafilters.** We now show that the conditions of Theorem 4.5 are met.

Theorem 4.6. *There is a $\mathscr{T}$-sequence, $\mathscr{U}_{\mathscr{T}}$, of remote ultrafilters.*

Proof. For each $n \in \omega$, choose a sequence $\{s(\omega, n, \ell) : \ell \in \omega\} \subset 2^{<\omega}$ so that

(1) for each ℓ, $t_n \subset s(\omega, n, \ell)$,
(2) the members of $\{[s(\omega, n, \ell)] : \ell \in \omega\}$ are pairwise disjoint,
(3) the sequence $\{[s(\omega, n, \ell)]_\omega : \ell \in \omega\}$ converges to the point $t(\omega, n)$ in the space 2^ω.

Claim 1. There is a sequence $\{s(\alpha, n, \ell) : \omega \leq \alpha \in \omega_1, n, \ell \in \omega\}$ satisfying

(1) $\{s(\alpha, n, \ell) : n, \ell \in \omega\} \subset \mathrm{Fn}(\alpha, 2)$,
(2) for all $n, \ell \in \omega$, $s(\omega, n, \ell) = s(\alpha, n, \ell) \restriction \omega$,
(3) the sequence $\{[s(\alpha, n, \ell)]_\alpha : \ell \in \omega\}$ converges to the point $t(\alpha, n)$ in the space 2^α,
(4) for all $\omega \leq \beta < \alpha$, there is an $m \in \omega$ such that for all $n > m$ and $\ell \in \omega$, $s(\beta, n, \ell) = s(\alpha, n, \ell) \restriction \beta$.

Proof of Claim: We prove the Claim by constructing the family by recursion on $\alpha \in \omega_1$. Assume that $\{s(\beta, n, \ell) : \omega \leq \beta < \alpha,\ \ n, \ell \in \omega\}$ have been defined satisfying the conditions (1)-(4) of the Claim. Let $\{\beta_j^\alpha : j \in \omega\}$ be an enumeration of $\alpha \setminus \omega$ (with repetitions allowed). For each $k \in \omega$, let α_k denote the maximum member of $\{\beta_j^\alpha : j \leq k\}$. Choose by our inductive hypotheses and Lemma 4.1(2), any strictly increasing sequence $\{m_k : k \in \omega\}$ chosen so that for all $n > m_k$, all $\ell \in \omega$ and all $j \leq k$, $s(\beta_j^\alpha, n, \ell) = s(\alpha_k, n, \ell) \restriction \beta_j^\alpha$ and $t(\beta_j^\alpha, n) = t(\alpha, n) \restriction \beta_j^\alpha$ (apply Lemma 4.1(2) on $\{\beta_j^\alpha : j \leq k\} \cup \{\alpha\}$). To see that m_k can indeed be chosen independently of ℓ, use the inductive assumption on convergence in (3).

For each $k \in \omega$ and $m_k < n \leq m_{k+1}$ we define the sequence $s(\alpha, n, \ell)$. First choose ℓ_0 large enough so that $t(\alpha_k, n) \restriction (\{\beta_j^\alpha : j \leq k\} \setminus \{\alpha_k\}) \subset s(\alpha_k, n, \ell)$ for all $\ell > \ell_0$ (by inductive assumption (3)). For $\ell \leq \ell_0$, let $s(\alpha, n, \ell) = s(\alpha_k, n, \ell)$. For $\ell > \ell_0$, define $s(\alpha, n, \ell)$ to be the union of $s(\alpha_k, n, \ell)$ and $t(\alpha, n) \restriction (\{\beta^\alpha_j : j \leq \ell\} \setminus \alpha_k)$. Since $\mathrm{dom}(s(\alpha_k, n, \ell)) \subset \alpha_k$, $s(\alpha_k, n, \ell) \in \mathrm{Fn}(\alpha, 2)$. Evidently, $s(\alpha_k, n, \ell) = s(\alpha, n, \ell) \restriction \alpha_k$ for all ℓ and since the family $\{[s(\alpha_k, n, \ell)]_{\alpha_k} : \ell \in \omega\}$ is pairwise disjoint, so too is the family $\{[s(\alpha, n, \ell)]_\alpha : \ell \in \omega\}$. We verify that $\{[s(\alpha, n, \ell]_\alpha : \ell \in \omega\}$ converges to $t(\alpha, n)$. Consider any finite $H \subset \alpha$. Since the family $\{[s(\alpha_k, n, \ell)]_{\alpha_k} : \ell \in \omega\}$ converges to $t(\alpha_k, n) = t(\alpha, n) \restriction \alpha_k$, there is a $\bar{\ell}$ large enough so that $t(\alpha_k, n) \restriction (H \cap \alpha_k) \subset s(\alpha_k, n, \ell)$ for all $\ell > \bar{\ell}$. We may choose $\bar{\ell}$ large enough so that $H \subset \{\beta^\alpha_j : j < \bar{\ell}\}$ and it then follows that $t(\alpha, n) \restriction H \subset s(\alpha, n, \ell)$ for all $\ell > \bar{\ell}$. We also note that for $j < k$ and $\ell \in \omega$, $s(\beta^\alpha_j, n, \ell) = s(\alpha_k, n, \ell) \restriction \beta^\alpha_j = s(\alpha, n, \ell) \restriction \beta^\alpha_j$.

It follows that for all $k \in \omega$, $n > m_k$ and $\ell \in \omega$ we have that $s(\beta^\alpha_k, n, \ell) = s(\alpha, n, \ell) \restriction \beta^\alpha_k$. This completes the proof of the Claim. □

Fix any $n, \ell \in \omega$ and $\alpha \in \omega_1$, and let $\mathscr{L}(\alpha, n, \ell)$ be the set of all $b \in \mathscr{M}_\alpha$ such that

(1) $b \subset [s(\alpha, n, \ell)]_\alpha$,
(2) b has measure greater than $\frac{2^\ell - 1}{2^\ell} \mu([s(\alpha, n, \ell)])$.

Claim 2. Let $n, \ell \in \omega$ and suppose that $b_j \in \mathscr{L}(\alpha_j, n, \ell)$ for each $j \leq k < \ell$ where $\omega \leq \alpha_0 \leq \ldots \leq \alpha_k \in \omega_1$ satisfies that $s(\alpha_j, n, \ell) = s(\alpha_k, n, \ell) \restriction \alpha_j$. Then $\bigcap_{j \leq k} \psi_{\alpha_j}(b_j) \cap [s(\alpha_k, n, \ell)]$ is not empty.

Proof of Claim: Of course $\psi_{\alpha_j}(b_j) \in \mathscr{M}^{\omega_1}_{\alpha_j}$ and is a subset of $[s(\alpha_j, n, \ell]$, that has measure greater than $\frac{2^\ell - 1}{2^\ell}$ times the measure of $[s(\alpha_j, n, \ell)]$. Hence it follows from the fact that $s(\alpha_j, n, \ell) = s(\alpha_k, n, \ell) \restriction \alpha_j$, that $\psi_{\alpha_j}(b_j) \cap [s(\alpha_k, n, \ell)]$ has measure greater than $\frac{2^\ell - 1}{2^\ell}$ times the measure of $[s(\alpha_k, n, \ell)]$. Let $J = |\mathrm{dom}(s(\alpha_k, n, \ell))|$ and since the measure of $\psi_{\alpha_j}(b_j) \cap [s(\alpha_k, n, j)]$ is greater than $\frac{2^\ell - 1}{2^{J+\ell}}$, it follows that the measure of $[s(\alpha_k, n, \ell)] \setminus \psi_{\alpha_j}(b_j)$ is less than $\frac{1}{2^{J+\ell}}$. Since $\frac{k+1}{2^{J+\ell}} \leq \frac{\ell}{2^{J+\ell}} < \frac{1}{2^J}$, it follows that $\bigcap_{j \leq k} b_j \cap [s(\alpha_k, n, \ell)]$ is not empty. □

Claim 3. For all $\alpha < \beta$ and $n, \ell \in \omega$ such that $s(\alpha, n, \ell) = s(\beta, n, \ell) \restriction \alpha$,

$$\psi_\alpha(\mathscr{L}(\alpha, n, \ell)) = \{\psi_\alpha(\mathrm{pr}_\alpha(b)) : b \in \psi_\beta(\mathscr{L}(\beta, n, \ell))\}.$$

Proof of Claim: Let $S_0 = \psi_\alpha(\mathscr{L}(\alpha, n, \ell))$ and $S_1 = \{\psi_\alpha(\mathrm{pr}_\alpha(b)) : b \in \psi_\beta(\mathscr{L}(\beta, n, \ell))\}$. Consider any $b \in \mathscr{L}(\alpha, n, \ell)$ and we show that $\psi_\alpha(b) \in S_1$. It suffices to show that $\psi_\alpha(b) \cap [s(\beta, n, \ell)]$ has measure greater than $\frac{2^\ell - 1}{2^\ell} \mu([s(\beta, n, \ell)]$. Since $\mu(b) > \frac{2^\ell - 1}{2^\ell} \mu([s(\alpha, n, \ell)])$, we may choose a finite family $\mathscr{A} \subset \{s \in Fn(\alpha, 2) : s(\alpha, n, \ell) \subset s\}$ so that $a = \bigcup\{[s] : s \in \mathscr{A}\} \in \mathscr{L}(\alpha, n, \ell)$ and $\frac{2^\ell - 1}{2^\ell} \mu([s(\alpha, n, \ell)]) + \mu(a \Delta b) < \mu(a)$. Since ψ_α is measure preserving, to show that $\psi_\alpha(b) \in S_1$, it suffices to show that $\psi_\alpha(a) \cap [s(\beta, n, \ell)]$ has measure greater than $\frac{2^\ell - 1}{2^\ell} \mu([s(\beta, n, \ell)])$. This fact now follows from the fact that for each $s \in \mathscr{A}$

$$\frac{\mu([s]_\alpha)}{\mu([s(\alpha, n, \ell)]_\alpha)} = \frac{1}{2^{|dom(s)| - J}} = \frac{\mu([s]_\beta \cap [s(\beta, n, \ell)]_\beta)}{\mu([s(\beta, n, \ell)]_\beta)}$$

where $J = |\operatorname{dom}(s(\alpha,n,\ell))|$. The reverse inclusion, namely that $S_2 \subset S_1$, follows similarly from the fact that for $s(\beta,n,\ell) \subset s \in \operatorname{Fn}(\beta,2)$,

$$\frac{\mu([s\restriction\alpha]_\alpha)}{\mu([s(\alpha,n,\ell)]_\alpha)} = \frac{1}{2^{|dom(s\restriction\alpha)|-J}} \geq \frac{1}{2^{|dom(s)|-J_\beta}} = \frac{\mu([s]_\beta)}{\mu([s(\beta,n,\ell)]_\beta)}$$

where $J_\beta = |\operatorname{dom}(s(\beta,n,\ell))|$. □

We define a filter $\mathscr{F}(\alpha,n)$ on $\mathscr{M}_\alpha$ for all $n\in\omega\leq\alpha\in\omega_1$. We let $b\in\mathscr{F}(\alpha,n)$ if $b\in\mathscr{M}_\alpha$ and there is an $\ell_b\in\omega$ such that $b\cap[s(\alpha,n,\ell)]\in\mathscr{L}(\alpha,n,\ell)$ for all $\ell>\ell_b$.

Recall the definitions in the second paragraph of subsection 4.3.

Claim 4. For each $n\in\omega\leq\alpha\leq\beta\in\omega_1$,

(1) $\mathscr{F}(\alpha,n)$ is a $\tilde{t}(\alpha,n)$-remote filter, and,
(2) if $t(\alpha,n)=t(\beta,n)\restriction\alpha$, then $\psi_\alpha(\mathscr{F}(\alpha,n))\subset\psi_\beta(\mathscr{F}(\beta,n))$,
(3) if $t(\alpha,n)=t(\beta,n)\restriction\alpha$, then $\{\psi_\alpha(pr_\alpha(b)) : b\in\psi_\beta(\mathscr{F}(\beta,n))\}=\psi_\alpha(\mathscr{F}(\alpha,n))$.

Proof of Claim: It follows from Claim 2 that $\psi_\alpha(\mathscr{F}(\alpha,n))$ is a filter on $\mathscr{M}_\alpha^{\omega_1}$, which of course ensures that $\mathscr{F}(\alpha,n)$ is a filter on $\mathscr{M}_\alpha$. For each ℓ_0, $\bigcup\{[s(\alpha,n,\ell)]_\alpha : \ell>\ell_0\}\in\mathscr{F}(\alpha,n)$, which ensures by (3) in Claim 1 that $g_\alpha(\mathscr{U})=t(\alpha,n)$ for all $\mathscr{F}(\alpha,n)\subset\mathscr{U}\in\operatorname{st}(\mathscr{M}_\alpha)$.

Suppose that $\mathscr{F}(\alpha,n)\subset\mathscr{U}\in\operatorname{st}(\mathscr{M}_\alpha)$. It remains to show that $\mathscr{U}$ is a remote $\tilde{t}(\alpha,n)$-filter. Let $D\subset\operatorname{st}(\mathscr{M}_\alpha)$ be nowhere dense and disjoint from $\tilde{t}(\alpha,n)$. For each $\ell\in\omega$, the set $D_\ell=D\cap[s(\alpha,n,\ell)]_\alpha$ is nowhere dense. Let $\mathscr{A}_\ell$ denote the set of $b\in\mathscr{M}_\alpha$ satisfying that $b\subset[s(\alpha,n,\ell)]_\alpha$ and $b^+\cap D_\ell=\emptyset$. In fact $\mathscr{A}_\ell$ is an ideal in the complete Boolean algebra $\mathscr{M}_\alpha$ and the join of $\mathscr{A}_\ell=[s(\alpha,n_\ell)]_\alpha$ since D_ℓ is nowhere dense. Therefore $\mu([s(\alpha,n,\ell)]_\alpha)$ is the least upper bound of the set $\{\mu(b) : b\in\mathscr{A}_\ell\}$. Choose $b_\ell\in\mathscr{A}_\ell$ such that $\mu(b_\ell)>\frac{2^\ell-1}{2^\ell}\mu([s(\alpha,n,\ell)]_\alpha)$.

The element $b=\bigcup_{\ell\in\omega}b_\ell\in\mathscr{F}(\alpha,n)$ and $b^+\cap D_\ell=b_\ell^+\cap D_\ell=\emptyset$ for all $\ell\in\omega$. Furthermore $b^+\subset\tilde{t}(\alpha,n)\cup\bigcup_{\ell\in\omega}b_\ell^+$, and so $b^+\cap D$ is empty.

Statements (2) and (3) of the Claim follow immediately from Claim 3. This completes the proof of Claim 4. □

We are now almost done with the proof. Recall that we aim at constructing a sequence such as in Definition 4.4. The following notion is convenient for the rest of the proof.

($\star$) For $\delta\in\omega_1$, a *$\mathscr{T}_\delta$-sequence of $\mathscr{F}$-ultrafilters* is a sequence $\{\mathscr{U}(\alpha,n) : \omega\leq\alpha<\delta, n\in\omega\}$ satisfying for each $\omega\leq\alpha\leq\beta<\delta$:
  (a) $\mathscr{F}(\alpha,n)\subset\mathscr{U}(\alpha,n)\in\operatorname{st}(\mathscr{M}_\alpha)$, hence $g_\alpha(\mathscr{U}(\alpha,n))=t(\alpha,n)$ for all $n\in\omega$,
  (b) for all but finitely many $n\in\omega$, $\psi_\alpha(\mathscr{U}(\alpha,n))$ is a subset of $\psi_\beta(\mathscr{U}(\beta,n))$.

Claim 5. If $\{\mathscr{U}(\alpha,n) : n\in\omega\leq\alpha<\delta\}$ is a $\mathscr{T}_\delta$-sequence of $\mathscr{F}$-ultrafilters for some $\delta\in\omega_1$, then there is a sequence $\{\mathscr{U}(\delta,n) : n\in\omega\}$ so that the sequence extends to a $\mathscr{T}_{\delta+1}$-sequence of $\mathscr{F}$-ultrafilters.

Proof of Claim: Fix a monotone increasing sequence $\{\alpha_k : k \in \omega\}$ cofinal in δ. By the assumptions (Lemma 4.1, Claim 4 and ($\star$)), there is a corresponding strictly increasing sequence $\{n_k : k \in \omega\}$ satisfying that, for all $n > n_k$:

(1) $t(\alpha_k, n) = t(\delta, n) \restriction \alpha_k$,
(2) $\psi_{\alpha_j}(\mathscr{U}(\alpha_j, n)) \subset \psi_{\alpha_k}(\mathscr{U}(\alpha_k, n))$ for all $j \leq k$,
(3) $\psi_{\alpha_k}(\mathscr{F}(\alpha_k, n)) \subset \psi_\delta(\mathscr{F}(\delta, n))$,
(4) $\{\psi_{\alpha_k}(pr_{\alpha_k}(b)) : b \in \psi_\delta(\mathscr{F}(\delta, n))\} \subset \psi_{\alpha_k}(\mathscr{F}(\alpha_k, n))$.

Fix any $k \in \omega$ and $n_k < n \leq n_{k+1}$. Since $\mathscr{F}(\alpha_k, n) \subset \mathscr{U}(\alpha_k, n)$, it follows from (4), that the family $\mathscr{G}(\delta, n) = \psi_{\alpha_k}(\mathscr{U}(\alpha_k, n)) \cup \psi_\delta(\mathscr{F}(\delta, n))$ has the finite intersection property in $\mathscr{M}_\delta^{\omega_1}$. We may therefore choose $\mathscr{U}(\delta, n) \in \mathrm{st}(\mathscr{M}_\delta)$ such that $\mathscr{F}(\delta, n) \subset \mathscr{U}(\delta, n)$ and $\mathscr{G}(\delta, n) \subset \psi_\delta(\mathscr{U}(\delta, n))$. Similarly, by (2), we have that $\psi_{\alpha_j}(\mathscr{U}(\alpha_j, n)) \subset \psi_\delta(\mathscr{U}(\delta, n))$ for all $j \leq k$. For this reason, it follows that for all $n > n_k$, $\psi_{\alpha_k}(\mathscr{U}(\alpha_k, n)) \subset \psi_\delta(\mathscr{U}(\delta, n))$.

Now let $\alpha < \delta$. Choose any $k \in \omega$ so that $\alpha \leq \alpha_k$. By the definition of $\mathscr{T}_\delta$-sequence of $\mathscr{F}$-ultrafilters, there is an $m \in \omega$ such that $\psi_\alpha(\mathscr{U}(\alpha, n)) \subset \psi_{\alpha_k}(\mathscr{U}(\alpha_k, n))$ for all $n > m$. Therefore, for all $n > \max(m, n_k)$, we have that $\psi_\alpha(\mathscr{U}(\alpha, n)) \subset \psi_\delta(\mathscr{U}(\delta, n))$. This completes the proof of Claim 5. □

Hence we are done. □

In view of Theorem 1.1, there is a $\mathfrak{c}$-OK set in $\mathbb{N}^*$ that maps irreducibly onto $\mathrm{st}(\mathscr{M}_{\omega_1})$. That map must be a homeomorphism by the fact that $\mathrm{st}(\mathscr{M}_{\omega_1})$ is extremally disconnected. Since a weak P-subset of a weak P-subset of a space is a weak P-set in that space, this completes the proof of Theorem 1.2.

5. Remarks

Let us first remark that $\mathrm{st}(\mathscr{M}_{\omega_1})$ satisfies the countable chain condition. This implies that the nontrivial copy of $\mathbb{N}^*$ in $\mathbb{N}^*$ that we get from the proof of Theorem 1.2, is not an ω_1-OK set (and hence not a $\mathfrak{c}$-OK set) by Kunen [28, 1.4]. This explains Question 1.1, which we repeat here for the sake of completeness.

Question 5.1. Is there a nowhere dense $\mathfrak{c}$-OK set in $\mathbb{N}^*$ that is homeomorphic to $\mathbb{N}^*$?

In the area 'special points in compact spaces', many problems motivated by Kunen's results are still open. An important one the affirmative solution of which would yield an 'honest' proof of Frolík's theorem from [20], is:

Question 5.2 (Dow [8])**.** If X is extremally disconnected and compact, does X contain a point x such that for any countable discrete subset $D \subset X \setminus \{x\}$ we have $x \notin \overline{D}$?

It was shown in [3], that the answer is affirmative for compact extremally disconnected spaces X satisfying $\pi\chi(X) = \pi w(X) \leq \mathfrak{c}$.

Another one, that goes in a completely different direction, is:

Question 5.3 (Kunen [30, 6.1])**.** Let X be a compact space of weight $\mathfrak{c}$ in which every nonempty G_δ has nonempty interior. Is there a weak P-point in X?

It is clear that this is true under CH.

These are just a couple of the many open problems of the type: is there a set (or point) in a given space which cannot be 'touched' by elements of a given collection of subsets in its complement?

Going in a different direction we suggest the following question.

Question 5.4. Is every closed subset of $\mathbb{N}^*$ homeomorphic to a subset that is a weak P-set?

We finish by stating and sketching the proof of another result in the same spirit as Theorem 1.2, which is an example of this and also illustrates the flexibility of the techniques used in this paper.

Theorem 5.1. *There is a copy X of $\mathbb{N}^*$ in $\mathbb{N}^*$ having the following properties:*

1. *There is a countable subset E contained in $\mathbb{N}^* \setminus X$ such that the closure of E contains X,*
2. *for every countable discrete subset F in $\mathbb{N}^* \setminus X$, the closure of F misses X.*

We recall that the proof in [13], roughly speaking, boils down to the following. In 2^{ω_1}, we attach a compatible ultrafilter in the absolute $E(2^{\omega_1})$ of 2^{ω_1}, to every node of an Aronszajn tree in 2^{ω_1}. In the proof of Theorem 1.1, we did the same thing in the Stone space of the measure algebra $\mathscr{M}_{\omega_1}$, with a subtle difference: the compatible ultrafilters were chosen to be remote points. And that last fact combined with Theorem 1.1 exactly made our construction work. The remote points in the proof were found by exploiting the natural measure on $\mathrm{st}(\mathscr{M}_{\omega_1})$. But this does not work in $E(2^{\omega_1})$ for example because in 2^{ω_1} there are many closures of countable discrete sets with positive measure. Hence the proofs of Theorems 1.2 and 5.1 are different when it comes to remote points. Instead of measures, we use the ideas in [5], [6] and [31, Lemma 1.2]. The reader who made it this far, will have no problem checking that the proof of Theorem 5.1 can be completed along the same lines of that of Theorem 1.2, once the following facts about remote points have been verified.

It will be convenient to say that a collection of nonempty subsets $\mathscr{F}$ of a space X is *remote* if for every nowhere dense subset D of X there exists $F \in \mathscr{F}$ such that $D \cap F = \emptyset$. Moreover, $\mathscr{F}$ is *n-linked* if for all $\mathscr{G} \in [\mathscr{F}]^{\leq n}$ we have $\bigcap \mathscr{G} \neq \emptyset$. If $\mathscr{F}$ is a collection of subsets of a set X, and $f : X \to Y$, then $f(\mathscr{F})$ denotes the collection $\{f(F) : F \in \mathscr{F}\}$.

Let $n \geq 1$, and let $\mathscr{F}$ be an n-linked collection of sets. For every $1 \leq i \leq n$, let

$$\mathscr{F}_{(i)} = \big\{ \bigcap \mathscr{G} : \mathscr{G} \in [\mathscr{F}]^{\leq i} \big\}.$$

Hence $\mathscr{F} = \mathscr{F}_{(1)} \subset \mathscr{F}_{(2)} \subset \cdots \subset \mathscr{F}_{(n)}$, and $\emptyset \notin \mathscr{F}_{(n)}$.

For the remainder of this section, X and Y are zero-dimensional compact spaces of countable weight, and $f : X \to Y$ is a continuous, open surjection. (X stands for 2^α and Y for 2^β for certain $\omega \leq \alpha \leq \beta < \omega_1$, and f for the projection $2^\beta \to 2^\alpha$.) Moreover, for some $n \geq 1$, $\mathscr{F}$ is an n-linked remote collection of clopen subsets of Y. Hence $Y \neq \emptyset$, and the same is true for X.

We will now show how to find an n-linked remote collection of clopen subsets of X which is compatible with f and $\mathscr{F}$. This is what we need to make the proof work.

Lemma 5.2. *For each nowhere dense subset D of X, there exists a nonempty clopen subset C of X such that $C \cap D = \emptyset$ and $f(C) \in \mathscr{F}$.*

Proof. Let D be any nowhere dense subset of X. We assume without loss of generality that D is closed. Put

$$E = \{y \in Y : f^{-1}(y) \subset D\}.$$

Then E is closed in Y (since f is open) and obviously nowhere dense. Hence we may pick $F \in \mathscr{F}$ such that $F \cap E = \emptyset$. Now, for every $y \in F$ we have that $f^{-1}(y) \setminus D \neq \emptyset$. Since F is clopen, for such y we may pick a clopen subset C_y in X such that $C_y \subset f^{-1}(F) \setminus D$ and $C_y \cap f^{-1}(y) \neq \emptyset$. The clopen cover $\{f(C_y) : y \in F\}$ of F has a finite subcover. Hence we may pick a finite $A \subset F$ such that $\bigcup_{y \in A} f(C_y) = F$. We conclude that the clopen set $C = \bigcup_{y \in A} C_y$ is as required. □

Put $\mathscr{U}^{(0)} = \{X\}$, and for every $1 \leq i \leq n$, let

$$\mathscr{U}^{(i)} = \{U \in \mathrm{CO}(X) : f(U) \in \mathscr{F}_{(i)}\}.$$

Observe that for $0 \leq i \leq n$, $\emptyset \notin \mathscr{U}^{(i)}$.

Corollary 5.3. *For each $0 \leq i \leq n-1$, $U \in \mathscr{U}^{(i)}$ and nowhere dense subset D in X, there is some $U' \in \mathscr{U}^{(i+1)}$ such that $U' \subset U \setminus D$.*

Proof. Fix $0 \leq i \leq n-1$, and $U \in \mathscr{U}^{(i)}$. Let $g = f \restriction U : U \to f(U)$. Then g is a continuous and open surjection. Moreover, $\mathscr{G} = \{F \cap f(U) : F \in \mathscr{F}\}$ is a remote $(n-i)$-linked collection of clopen subsets of $f(U)$. If $D \subset X$ is nowhere dense, then $D \cap U$ is nowhere dense in U (as well as in X). Hence by Lemma 5.2, there exist a clopen subset U' in U and an element $F \in \mathscr{F}$ such that $U' \subset U \setminus D$ and $g(U') = f(U') = F \cap f(U)$. Hence U' is as required. □

Lemma 5.4. *There is a remote collection $\mathscr{G}$ of clopen subsets of X such that for every $m \leq n$ and $\mathscr{M} \in [\mathscr{G}]^m$, $f(\bigcap \mathscr{M})$ contains an element from $\mathscr{F}_{(m)}$. Hence $\mathscr{G}$ is n-linked.*

Proof. Let $\mathscr{D}$ be the collection of all nowhere dense subsets of X, and fix $D \in \mathscr{D}$ for a while.

Recall that $\mathscr{U}^{(0)} = \{X\}$. Let $1 \leq i \leq n$. Enumerate $\mathscr{U}^{(i)}$ as $\{U^i_k : k \in \omega\}$ (repetitions permitted). Moreover, for an arbitrary nowhere dense set $D \subset X$, put

$$H(D, i) = \{k \in \omega : U^i_k \cap D = \emptyset\}.$$

By Corollary 5.3, $H(D, 1) \neq \emptyset$. Let $\kappa(D, 1) = \min H(D, 1)$, and, for $2 \leq i \leq n$, define $\kappa(D, 2), \ldots, \kappa(D, n)$ by recursion, as follows:

$$\kappa(D, i) = \min\{k \in \omega : (\forall\, s \leq \kappa(D, i-1))(\exists\, t \leq k)(t \in H(D, i) \,\&\, U^i_t \subset U^{i-1}_s)\}.$$

Again by Corollary 5.3, $\kappa(D, i)$ is well-defined. Finally, put

$$F(D) = \bigcup_{i=1}^{n} \bigcup \{U^i_k : k \leq \kappa(D, i) \text{ and } k \in H(D, i)\}.$$

Observe that $F(D)$ is clopen, misses D, and is nonempty.

Claim 1. If $\mathscr{L}$ is a subfamily of $\mathscr{D}$ of cardinality e, where $1 \le e \le n$. then $\bigcap_{L\in\mathscr{L}} F(L) \supseteq U_l^e$ for some $l \le \max\{\kappa(L,e) : L \in \mathscr{L}\}$.

We prove this by induction on e. The case $e = 1$ is trivial, so assume the claim to be proven for all $1 \le i < j$, where $j \le n$. Let $\mathscr{L}$ be a subfamily of $\mathscr{D}$ of cardinality j. Put

$$\kappa = \max\{\kappa(L, j-1) : L \in \mathscr{L}\},$$

and take $L_0 \in \mathscr{L}$ such that $\kappa = \kappa(L_0, j-1)$. Put $\mathscr{L}' = \mathscr{L} \setminus \{L_0\}$. By our inductive assumption,

$$\bigcap_{L\in\mathscr{L}'} F(L) \supseteq U_{l'}^{j-1}$$

for some $l' \le \max\{\kappa(L, j-1) : L \in \mathscr{L}'\}$. Since

$$l' \le \max\{\kappa(L, j-1) : L \in \mathscr{L}'\} \le \kappa(L_0, j-1),$$

there is some $l \le \kappa(L_0, j)$ such that $U_l^j \subset U_{l'}^{j-1}$. Therefore, $\bigcap_{L\in\mathscr{L}} F(L) \supseteq U_l^j$ and since $l \le \kappa(L_0, j) \le \max\{\kappa(L, j) : L \in \mathscr{L}\}$, this completes the inductive proof.

Hence $\mathscr{G} = \{F(D) : D \in \mathscr{D}\}$ is as required. □

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