A ZERO-DIMENSIONAL F-SPACE THAT IS NOT STRONGLY ZERO-DIMENSIONAL

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Abstract. We present an example of a zero-dimensional F-space that is not strongly zero-dimensional.

Introduction

In this paper we present an example of a zero-dimensional F-space that is not strongly zero-dimensional. We recall that a space is *zero-dimensional* if it is a T_1-space and its clopen subsets form a base for the topology. The fastest way to define a space to be *strongly zero-dimensional* is by demanding that its Čech-Stone compactification is zero-dimensional.

The question whether zero-dimensionality implies strong zero-dimensionality has a long history, a summary of which can be found in [3, Section 6.2]. There are by now many examples of zero-dimensional spaces that are not strongly zero-dimensional, even metrizable ones, see [12], but the authors are not aware of an F-space of this nature.

The question whether there is a F-space example was making the rounds already in the 1980s but it seems to have been asked explicitly only a few years ago on MathOverFlow, see [11]. Recently Ali Reza Olfati raised the question with the first author in a different context.

In section 1 we give proper definitions of the notions mentioned above and indicate why it may seem reasonable, but also illusory, to expect that zero-dimensional F-spaces are strongly zero-dimensional.

In section 2 we construct the example and in section 3 we discuss some variations; the example can have arbitrary large covering dimension and its Čech-Stone remainder can be an indecomposable continuum.

1. Preliminaries

In the introduction we defined zero-dimensional spaces as T_1-spaces in which the clopen sets constitute a base for the open sets and strong zero-dimensionality by requiring that the Čech-Stone compactification is zero-dimensional.

The latter is a characterization of strong zero-dimensionality. The real definition is akin to the large inductive dimension: a Tychonoff space X is strongly zero-dimensional if any two completely separated sets are separated by a clopen set, that is, if A and B are such that there is a continuous function $f : X \to [-1, 1]$ with $f[A] = \{-1\}$ and $f[B] = \{1\}$ then there is a clopen set C such that $A \subseteq C$ and $C \cap B = \emptyset$. One could reformulate the latter conclusion as: there is a continuous function $c : X \to \{-1, 1\}$ such that $c[A] = \{-1\}$ and $c[B] = \{1\}$. It is not hard to show that this is equivalent to βX being zero-dimensional.

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Furthermore, for normal spaces strong zero-dimensionality is characterized by the 'normal' sounding "disjoint closed sets are contained in complementary clopen sets".

There are many characterizations of F-spaces, see [4, Theorem 14.25], each of which deserves to be taken as the definition but we take the one that at first glance seems quite close to strong zero-dimensionality; it is number (5) in the theorem referred to above. A Tychonoff space X is an F-space iff for every continuous function $f : X \to \mathbb{R}$ there is another continuous function $k : X \to \mathbb{R}$ with the property that $f = k \cdot |f|$; so k is constant on the sets $\{x : f(x) > 0\}$ and $\{x : f(x) < 0\}$ with values 1 and -1 respectively. Although k does seem to act like the function c in our definition of strong zero-dimensionality, it does not.

In fact there are (compact) connected F-spaces, for example $\beta\mathbb{R}^+ \setminus \mathbb{R}^+$, where $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geqslant 0\}$, see [4, 14.27] or [5]. In such spaces the function k takes on all values in the interval $[-1, 1]$ on the set $\{x : f(x) = 0\}$, which apparently need not be as thin as we have come to expect from Calculus; in an F-space sets like $\{x : f(x) > 0\}$ and $\{x : f(x) < 0\}$ are actually very far apart.

Our notation is standard, see [3] and [4] for topological notions, and [9] for Set Theory.

2. A zero-dimensional F-space that is not strongly zero-dimensional

The construction in this section is inspired by an answer to a question on MathOverFlow, see [1], which in turn was inspired by Dowker's example M in [2]. The latter is a subspace of $\omega_1 \times [0, 1]$; the example on MathOverFlow is a quotient of $\omega_1 \times \mathbb{A}$, where $\mathbb{A}$ is Alexandroff's split interval.

We replace the ordinal space ω_1 by the G_δ-modification of the ordinal space ω_2, which we denote $(\omega_2)_\delta$; likewise $(\omega_2 + 1)_\delta$ denotes the G_δ-modification of $\omega_2 + 1$. We replace $\mathbb{A}$ by the split interval over a suitable ordered continuum.

We shall use an ordered continuum K with a dense subset D that can be enumerated as $\langle d_\alpha : \alpha \in \omega_2\rangle$ in such a way that every tail set $T_\alpha = \{d_\beta : \beta \geqslant \alpha\}$ is dense in K.

Example 1. If CH fails then we can take $K = [0, 1]$ and, like Dowker did, choose $\aleph_2$ many distinct cosets of $\mathbb{Q}$, say $\{\mathbb{Q}_\alpha : \alpha \in \omega_2\}$, and enumerate their union D as $\langle d_\alpha : \alpha \in \omega_2\rangle$ in such a way that $\langle d_{\omega\cdot\alpha+n} : n \in \omega\rangle$ enumerates $\mathbb{Q}_\alpha \cap (0, 1)$.

Example 2. For a ZFC example let M be the linearly ordered sum $\omega_2^\star + \{\mathbf{0}\} + \omega_2$, where $\omega_2^\star$ denotes ω_2 with its order reversed. Following [7] we let $L = M_{\mathbf{0}}(\omega)$, that is, the set $\{x \in M^\omega : \{m : x_n \neq \mathbf{0}\}$ is finite$\}$, ordered lexicographically. It is elementary to verify that the linear order is dense, in fact every interval has cardinality $\aleph_2$, and has no smallest or largest element.

We let K be the Dedekind completion of L (see [8, Kap. IV, § 5]), that is, the set of initial segments that have no maximum (including $\emptyset$ and L), ordered by inclusion. Then K is an ordered continuum and the set L itself serves as the desired dense set, under any enumeration.

We need the following Lemma, which is a variation of a result of Van Douwen, see [3, Problem 3.12.20.(c)].

Lemma 2.1. *Let X be a compact Hausdorff space. The product $(\omega_2)_\delta \times X$ is C-embedded in $(\omega_2 + 1)_\delta \times X$.*

Proof. Let $f : (\omega_2)_\delta \times X \to \mathbb{R}$ be continuous.

Take $\alpha \in \omega_2$ of cofinality $\aleph_1$. For every $x \in X$ and $n \in \omega$ one can find $\beta(x,n) < \alpha$ and an open set $U(x,n)$ in X such that $x \in U(x,n)$ and

$$f\bigl[(\beta(x,n),\alpha] \times U(x,n)\bigr] \subseteq \bigl(f(\alpha,x) - 2^{-n}, f(\alpha,x) + 2^{-n}\bigr)$$

By compactness we can take a finite subcover $\{U(x,n) : x \in F_n\}$ of the cover $\{U(x,n) : x \in X\}$. Let $\beta_n = \max\{\beta(x,n) : x \in F_n\}$, then for all $x \in X$ and $\gamma \in (\beta_n, \alpha]$ we have $\bigl|f(\gamma,x) - f(\alpha,x)\bigr| < 2^{-n+1}$.

Next let $\beta_\alpha = \sup\{\beta_n : n \in \omega\}$, then $\beta_\alpha < \alpha$ and f is constant on each horizontal line $(\beta_\alpha, \alpha] \times \{x\}$.

The Pressing-Down Lemma now gives us a single β such that f is constant on $(\beta, \omega_2) \times \{x\}$ for all x. Those constant values give us our continuous extension of f to $(\omega_2 + 1)_\delta \times X$. □

The rest of the section is devoted to the construction of our F-space.

Split intervals. Using the continuum K and the dense set $\{d_\alpha : \alpha \in \omega_2\}$ we create a sequence $\langle K_\alpha : \alpha \leqslant \omega_2\rangle$ of ordered compacta, as follows:

$$K_\alpha = \{\langle x, i\rangle \in K \times 2 : \text{if } x \notin T_\alpha \text{ then } i = 0\}$$

ordered lexicographically (reading from left to right). Thus K_α is a split interval over K, where all points d_β with $\beta \geqslant \alpha$ are split in two; if $\alpha = \omega_2$ then no points are split and K_{ω_2} is just K itself.

There are obvious maps $q_{\alpha,\beta} : K_\alpha \to K_\beta$ when $\alpha < \beta$, defined by

$$q_{\alpha,\beta}(x,i) = \langle x, 0\rangle \text{ when } x \notin \{d_\gamma : \gamma \geqslant \beta\}$$
$$q_{\alpha,\beta}(d_\gamma,i) = \langle d_\gamma, i\rangle \text{ when } \gamma \geqslant \beta.$$

We abbreviate the maps $q_{0,\alpha}$ by q_α.

If $\alpha < \omega_2$ then K_α is zero-dimensional. Here is where we use that every tail set T_α is dense in K. This implies that the family $\mathcal{B}_\alpha$ of all clopen intervals of the form $\bigl[\min K, \langle e, 0\rangle\bigr]$, $\bigl[\langle d, 1\rangle, \langle e, 0\rangle\bigr]$, and $\bigl[\langle d, 1\rangle, \max K\bigr]$, where $d, e \in T_\alpha$, is base for the topology of K_α. As K_α is compact it is strongly zero-dimensional as well.

For later use: the intervals in $\mathcal{B}_\alpha$ belong to $\mathcal{B}_\beta$ when $\beta \leqslant \alpha$ (when suitably interpreted) and if $I \in \mathcal{B}_\alpha$ is such an interval then it satisfies $I = q_{\beta,\alpha}^{\leftarrow}\bigl[q_{\beta,\alpha}[I]\bigr]$ whenever $\beta \leqslant \alpha$.

Using compactifications. To get to our F-space we take, for every $\alpha \leqslant \omega_2$, the Čech-Stone compactification $\beta(\omega \times K_\alpha)$ of the product $\omega \times K_\alpha$; we let $\mathbb{K}_\alpha$ denote this compactification and X_α denotes the remainder $(\omega \times K_\alpha)^*$. The maps $q_{\alpha,\beta}$ induce maps from $\mathbb{K}_\alpha$ to $\mathbb{K}_\beta$ when $\alpha < \beta$; we denote these by $\boldsymbol{q}_{\alpha,\beta}$, and $\boldsymbol{q}_\alpha = \boldsymbol{q}_{0,\alpha}$ of course.

If $\alpha < \omega_2$ then the product $\omega \times K_\alpha$ is strongly zero-dimensional because K_α is; this implies that $\mathbb{K}_\alpha$ and X_α are zero-dimensional and hence, by compactness, strongly zero-dimensional as well. Furthermore, by [4, 14.27], every X_α is an F-space, including for $\alpha = \omega_2$.

We consider the product $(\omega_2)_\delta \times \mathbb{K}_0$ and the union

$$\mathbb{K} = \bigcup\bigl\{\{\alpha\} \times \mathbb{K}_\alpha : \alpha < \omega_2\bigr\}$$

as well as $(\omega_2 + 1)_\delta \times \mathbb{K}_0$ and $\mathbb{K}^+ = \mathbb{K} \cup (\{\omega_2\} \times \mathbb{K}_{\omega_2})$.

Our example will be the union of the remainders:

$$X = \bigcup\bigl\{\{\alpha\} \times X_\alpha : \alpha < \omega_2\bigr\}$$

and we also use $X^+ = X \cup (\{\omega_2\} \times X_{\omega_2})$.

A quotient map and the topology. We define $\boldsymbol{q} : (\omega_2+1)_\delta \times \mathbb{K}_0 \to \mathbb{K}^+$ by combining the maps $\boldsymbol{q}_\alpha$:

$$\boldsymbol{q}(\alpha, x) = \bigl\langle \alpha, \boldsymbol{q}_\alpha(x)\bigr\rangle$$

We give $\mathbb{K}^+$ the quotient topology determined by $\boldsymbol{q}$ and the product topology on $(\omega_2+1)_\delta \times \mathbb{K}_0$. We show that $\boldsymbol{q}$ is a perfect map. The fibers of $\boldsymbol{q}$ are clearly compact so we must show that $\boldsymbol{q}$ is closed.

To begin note that for each α the set $\{\alpha\} \times \mathbb{K}_\alpha$ is closed and the map $\boldsymbol{q}_\alpha : \mathbb{K}_0 \to \mathbb{K}_\alpha$ is a closed map, so that the quotient topology on $\{\alpha\} \times \mathbb{K}_\alpha$ is its normal topology. Also, if α has countable cofinality then $\{\alpha\} \times \mathbb{K}_0$ is clopen in the product, hence so is $\{\alpha\} \times \mathbb{K}_\alpha$ in $\mathbb{K}^+$.

Next let α be of uncountable cofinality, take $x \in \mathbb{K}_\alpha$ and an open set O in $(\omega_2+1)_\delta \times \mathbb{K}_0$ such that $\boldsymbol{q}^\leftarrow(\alpha, x) = \{\alpha\} \times \boldsymbol{q}_\alpha^\leftarrow(x) \subseteq O$. By compactness there are an open set V in $\mathbb{K}_0$ and $\beta < \alpha$ such that

$$\{\alpha\} \times \boldsymbol{q}_\alpha^\leftarrow(x) \subseteq (\beta, \alpha] \times V \subseteq O$$

Because $\boldsymbol{q}_\alpha : \mathbb{K}_0 \to \mathbb{K}_\alpha$ is closed there is an open set U in $\mathbb{K}_\alpha$ such that $\boldsymbol{q}_\alpha^\leftarrow[U] \subseteq V$. Then $(\beta, \alpha] \times \boldsymbol{q}_\alpha^\leftarrow[U]$ is an open subset of $(\omega_2+1)_\delta \times \mathbb{K}_0$. For $\gamma \in (\beta, \alpha)$ we have $\boldsymbol{q}_\alpha = \boldsymbol{q}_{\gamma,\alpha} \circ \boldsymbol{q}_\gamma$, hence $\boldsymbol{q}_\alpha^\leftarrow[U] = \boldsymbol{q}_\gamma^\leftarrow\bigl[\boldsymbol{q}_{\gamma,\alpha}^\leftarrow[U]\bigr]$.

It follows that $\boldsymbol{q}^\leftarrow[W] = (\beta, \alpha] \times \boldsymbol{q}_\alpha^\leftarrow[U]$, where

$$W = \bigcup\bigl\{\{\gamma\} \times \boldsymbol{q}_{\gamma,\alpha}^\leftarrow[U] : \beta < \gamma \leqslant \alpha\bigr\}$$

The set W is therefore open and $\boldsymbol{q}^\leftarrow[W] \subseteq O$.

This argument also shows that $\mathbb{K}$ is zero-dimensional because if $\alpha < \omega_2$ the set U can be taken to be a clopen set and the resulting set W is clopen as well.

Thus far we have topologized $\mathbb{K}^+$ and hence X^+ and we have shown that X is zero-dimensional. We now turn to showing that X^+ is an F-space and that X is C-embedded in X^+. This will show that $\beta X = \beta X^+$, hence X is an F-space as well (by [4, 14.25]) but not strongly zero-dimensional because the one-dimensional space X_{ω_2} is a subspace of βX; we establish the one-dimensionality of X_{ω_2} in the next section.

C-embedding. To show that X is C-embedded in X^+ we let $f : X \to \mathbb{R}$ be continuous and apply the proof of Lemma 2.1 to $f \circ \boldsymbol{q} : (\omega_2)_\delta \times X_0 \to \mathbb{R}$ to find an $\alpha < \omega_2$ such that $f \circ \boldsymbol{q}$ is constant on $(\alpha, \omega_2) \times \{x\}$ for all $x \in X$, which then determines the (unique) extension $g : (\omega_2+1)_\delta \times X_0 \to \mathbb{R}$ of $f \circ \boldsymbol{q}$.

We show that $g(\omega_2, x) = g(\omega_2, y)$ whenever $\boldsymbol{q}_{\omega_2}(x) = \boldsymbol{q}_{\omega_2}(y)$; for then g determines a continuous extension of f to X^+. We assume $x \neq y$ of course and take disjoint neighbourhoods U and V of x and y in $\mathbb{K}_0$.

Using the compactness of K_0 we find two sequences $\langle \mathcal{I}_n : n \in \omega\rangle$ and $\langle \mathcal{J}_n : n \in \omega\rangle$ of finite subfamilies of $\mathcal{B}_0$ such that the clopen sets $I = \bigcup\bigl\{\{n\} \times \bigcup \mathcal{I}_n : n \in \omega\bigr\}$ and $J = \bigcup\bigl\{\{n\} \times \bigcup \mathcal{J}_n : n \in \omega\bigr\}$ satisfy

- $I \in x$ and $J \in y$ (x and y are ultrafilters of closed sets), and
- $I \subseteq U$ and $J \subseteq V$.

For each n let E_n be the set of points in K that occur as first coordinates of endpoints of one of the intervals in $\mathcal{I}_n$ and $\mathcal{J}_n$. The union, E, of these sets is countable. Therefore there is a $\beta \geqslant \alpha$ such that $E \cap T_\beta = \emptyset$. This means that for $\gamma \geqslant \beta$ the restriction $q_{\gamma,\omega_2} \restriction E$ is injective.

Because $\boldsymbol{q}_{\omega_2}(x) = \boldsymbol{q}_{\omega_2}(y)$ the intersection of $q_{\omega_2}[I]$ and $q_{\omega_2}[J]$ is not compact. For every n and intervals $A \in \mathcal{I}_n$ and $B \in \mathcal{J}_n$ the intersection of $\boldsymbol{q}[A]$ and $\boldsymbol{q}[B]$ is contained in E_n. Therefore $\boldsymbol{q}[I] \cap \boldsymbol{q}[J]$ is contained in $F = \bigcup\bigl\{\{n\} \times E_n : n \in \omega\bigr\}$ and hence the common value of $\boldsymbol{q}(x)$ and $\boldsymbol{q}(y)$ belongs to $\operatorname{cl} F$. As the maps q_{γ,ω_2} are injective on E for $\gamma \geqslant \beta$, so are the maps $\boldsymbol{q}_{\gamma,\omega_2}$ on $X_\gamma \cap \operatorname{cl} F$ whenever $\gamma \geqslant \beta$.

It follows that for all $\gamma \geqslant \beta$ we have $\boldsymbol{q}_\gamma(x) = \boldsymbol{q}_\gamma(y)$ and therefore

$$g(\gamma, x) = f(\gamma, \boldsymbol{q}_\gamma(x)) = f(\gamma, \boldsymbol{q}_\gamma(y)) = g(\gamma, y)$$

and this implies $g(\omega_2, x) = g(\omega_2, y)$, as desired.

F-space. To see that X^+ is an F-space let $f : X^+ \to \mathbb{R}$ be continuous. We seek a continuous function $k : X^+ \to \mathbb{R}$ such that $f = k \cdot |f|$.

As in the proof above we take $\alpha < \omega_2$ such that $f \circ \boldsymbol{q}$ is constant on all horizontal lines $(\alpha, \omega_2] \times \{x\}$.

Since X_{ω_2} is an F-space we get a continuous function $g : X_{\omega_2} \to \mathbb{R}$ such that $f(\omega_2, x) = g(x) \cdot \bigl|f(\omega_2, x)\bigr|$.

For all $\beta > \alpha$ we define k_{ω_2} on $\{\beta\} \times X_\beta$ by $k_{\omega_2}(\gamma, x) = g(\boldsymbol{q}_{\gamma,\omega_2}(x))$, and k^+ on $\{\gamma\} \times X_0$ by $k^+(\gamma, x) = g(\boldsymbol{q}_{\omega_2}(x))$. Then k^+ is continuous and $k^+ = k_{\omega_2} \circ \boldsymbol{q}$ on $(\alpha, \omega_2] \times X_0$, so that k_{ω_2} is continuous as well. Rename α as β_{ω_2}.

Now repeat this argument for every α of cofinality $\aleph_1$. First find $\beta_\alpha < \alpha$, as in the proof of Lemma 2.1, such that $f \circ \boldsymbol{q}$ is constant on $(\beta_\alpha, \alpha] \times \{x\}$ for all x, find a g on X_α and define k_α on $\{\gamma\} \times X_\gamma$, for $\gamma \in (\beta_\alpha, \alpha]$ as above by $k_\alpha(\gamma, x) = g(\boldsymbol{q}_{\gamma,\alpha}(x))$.

Finally, for every α of countable cofinality take $k_\alpha : \{\alpha\} \times X_\alpha \to \mathbb{R}$ such that $f(\alpha, x) = k_\alpha(\alpha, x) \cdot \bigl|f(\alpha, x)\bigr|$ for all x.

Since $(\omega_2{+}1)_\delta$ is Lindelöf there is a countable subset C of ω_2 consisting of ordinals of cofinality $\aleph_1$ such that the interval $(\beta_{\omega_2}, \omega_2]$ together with $\bigl\{(\beta_\alpha, \alpha] : \alpha \in C\bigr\}$ covers all but countably many points of $\omega_2 + 1$. From this it is easy to construct a pairwise disjoint clopen cover of $(\omega_2 + 1)_\delta$ and combine the various k_α into one continuous function.

3. Some variations and questions

The construction of our main example admits various variations.

Arbitrarily large covering dimension. To get a zero-dimensional F-space of a prescribed covering dimension n everywhere in the main construction replace K_α by K_α^n. Then $\mathbb{K}_{\omega_2} = \beta(\omega \times K^n)$. By the main result of [10] we have $\dim K^n = n$. The proof of this establishes that the pairs of opposite faces of this 'n-cube' form an essential family. To elaborate: write $\min K = 0$ and $\max K = 1$ and for $i \in n$ put $A_i = \{x \in K^n : x_i = 0\}$ and $B_i = \{x \in K^n : x_i = 1\}$. Then for every sequence $\langle L_i : i \in n\rangle$ of partitions of K^n, with L_i between A_i and B_i, the intersection $\bigcap_{i\in n} L_i$ is nonempty. By the Theorem on Partitions, [3, Theorem 7.2.15], this establishes $\dim K^n \geqslant n$. In addition [10] establishes the inequality $\operatorname{Ind} K^n \leqslant n$. We conclude that $\dim K^n = \operatorname{ind} K^n = \operatorname{Ind} K^n = n$.

To see that $\dim X_{\omega_2} = n$ as well, we consider the projection map $\pi : \omega \times K^n \to \omega$ and its extension $\beta\pi$. In [5, Section 2] it is shown that the components of $\mathbb{K}_{\omega_2}$ are exactly the fibers $\beta\pi^{\leftarrow}(u)$ for $u \in \beta\mathbb{N}$.

Next we let $\mathbb{A}_i = \operatorname{cl}(\omega \times A_i)$ and $\mathbb{B}_i = \operatorname{cl}(\omega \times B_i)$ for $i \in n$. An elementary topological argument will show that for every $u \in \mathbb{N}^*$ the intersections of the $\mathbb{A}_i$ and $\mathbb{B}_i$ with $\beta\pi^{\leftarrow}(u)$ form an essential family. This, together with the equality $\dim \beta Z = \dim Z$ ([3, Theorem 7.1.17]), shows that every component of X_{ω_2} has covering dimension n.

To see that in this case also $\dim X = n$ we first observe that $\dim X = \dim \beta X \geqslant \dim X_{\omega_2} = n$. To get the opposite inequality we let $\mathcal{U}$ be is a finite open cover of X^+. Its restriction to X_{ω_2} has a finite closed refinement of order $n + 1$, which can be expanded to a finite family $\mathcal{V}$ of open sets that also has order $n{+}1$, covers X_{ω_2}, and refines $\mathcal{U}$. The argument given below that $\beta X \setminus X_{\omega_2}$ is zero-dimensional produces an α such that $\bigcup_{\beta>\alpha}\{\beta\} \times X_\beta \subseteq \bigcup \mathcal{V}$. The rest of the space, $\bigcup_{\beta\leqslant\alpha}\{\beta\} \times X_\beta$, is strongly zero-dimension so the restriction of $\mathcal{U}$ to this clopen set has a disjoint open refinement.

This proof will also work if one takes K_α^ω everywhere, in which case every component of X_{ω_2} will be infinite-dimensional.

Most of the other arguments in Section 2 do not rely on the particular structure of the K_α, except the proof of C-embedding.

One still obtains finite sets E_m of points in K that occur as end points of intervals used to create clopen sets in $\{m\} \times K_0^n$ and hence in $\omega \times K_0^n$. One also still obtains a $\beta \geqslant \alpha$ with $E \cap T_\beta = \emptyset$.

The set F is replaced by $\bigcup\bigl\{\{m\} \times G_m : m \in \omega\bigr\}$, where G_m is the grid on K^n defined by $\{x : (\exists i \in n)(x_i \in E_m)\}$. Then $\boldsymbol{q}_{\gamma,\omega_2}$ is injective on $\{\gamma\} \times \operatorname{cl} F$ for all $\gamma \geqslant \beta$.

In case $n = \omega$ there is for every m a natural number k_m such that the supports of the clopen rectangles used in $\{m\} \times K_0^\omega$ are contained in k_m. In that case one takes $G_m = \{x : (\exists i \in k_m)(x_i \in E_m)\}$.

Indecomposability. It is possible to make X_{ω_2} an indecomposable continuum.

To this end we take a preliminary quotient of every $\omega \times K_\alpha$ by identifying $\langle n, 1\rangle$ and $\langle n+1, 0\rangle$ for every n (we still use $0 = \min K$ and $1 = \max K$). The result is an infinite string of copies of K_α and in case $\alpha = \omega_2$ the result is a connected ordered space $\mathbb{L}$, with a minimum, but no maximum. From a distance it looks like the half line $\mathbb{H} = [0, \infty)$ in $\mathbb{R}$, with every interval $[n, n+1]$ replaced by K.

The proof that $\mathbb{H}^*$ is an indecomposable continuum, see [5, Section 4], goes through without any changes to show that $X_{\omega_2} = \mathbb{L}^*$ is indecomposable as well.

The proofs that X is zero-dimensional and C-embedded in the F-space X^+ are not affected by these identifications.

We note that if we assume $\neg\mathsf{CH}$ and use $K = [0, 1]$ then X_{ω_2} is actually equal to the remainder $\mathbb{H}^*$ of the half line.

Local compactness. In all variations the space X_{ω_2} is the sole cause of the failure of strong zero-dimensionality.

To see this note first that the sets $F_\alpha = \bigcup_{\beta>\alpha}\{\beta\} \times X_\beta$ form a neighbourhood base at $\{\omega_2\} \times X_{\omega_2}$. Indeed if $f : X^+ \to \mathbb{R}$ is continuous and equal to zero on $\{\omega_2\} \times X_{\omega_2}$ then there is an $\alpha < \omega_2$ such that f is constant and equal to zero on F_α and the latter is a clopen subset of X^+.

Furthermore the complement of such a set, $I_\alpha = \bigcup_{\beta\leqslant\alpha}\{\beta\} \times X_\beta$, is strongly zero-dimensional. The fastest way to see this is to note that the product $(\alpha+1)_\delta \times K_0$ is Lindelöf as a product of a compact and a Lindelöf space. Therefore I_α is Lindelöf as well. By [3, Theorem 6.2.7] the zero-dimensional Lindelöf space I_α is strongly zero-dimensional.

Therefore the union $Z = \bigcup_{\alpha\in\omega_2} \operatorname{cl} I_\alpha$ is an open cover of $\beta X \setminus X_{\omega_2}$ by compact zero-dimensional sets and hence zero-dimensional.

It follows that Z is a locally compact zero-dimensional F-space that is not strongly zero-dimensional.

Questions. Our examples have weight $\aleph_2^{\aleph_0}$, so under CH the ZFC example cannot be embedded into $\mathbb{N}^*$. We do not know whether it can be embedded if CH fails. In fact we do not know the answer to the following question, which has been asked before but bears repeating often.

Question 1. Is there a subspace of $\mathbb{N}^*$ that is not strongly zero-dimensional?

Weight $\aleph_1$. It is well-known, and easy to see, that every space of cardinality less than $\mathfrak{c}$ is strongly zero-dimensional.

A similar phenomenon can be observed among F-spaces.

If X is an F-space and $f : X \to \mathbb{R}$ is continuous then for every $r \in \mathbb{R}$ the closures of $\{x : f(x) < r\}$ and $\{x : f(x) > r\}$ are disjoint and the complement of the union of these closures is an open set, O_r say. It follows that if the cellularity of X is less than $\mathfrak{c}$ then there will be many r such that $O_r = \emptyset$. For such r the closures of $\{x : f(x) < r\}$ and $\{x : f(x) > r\}$ would be complementary clopen sets. We find that F-spaces of cellularity less than $\mathfrak{c}$ are automatically strongly zero-dimensional. In case its cellularity is countable an F-space is even extremally disconnected, which means that disjoint open sets have disjoint closures.

What our space leaves unanswered is what happens for F-spaces of weight $\aleph_1$. Of course if $\aleph_1 < \mathfrak{c}$ then the comments above show that there is nothing more to investigate. Therefore we should assume the Continuum Hypothesis in order to obtain non-trivial questions and results.

It has been a rule-of-thumb under the assumption of CH that F-spaces of weight $\aleph_1$ show many parallels with separable metrizable spaces. In [6] one finds versions for compact F-spaces of weight $\aleph_1$ of some well-known theorems for compact metrizable spaces. In particular that the three main dimension functions coincide on this class.

We ask whether this holds without the compactness condition, assuming CH of course.

Question 2. Is every zero-dimensional F-space of weight $\aleph_1$ strongly zero-dimensional?

And more generally.

Question 3. Does the equality $\dim X = \operatorname{ind} X = \operatorname{Ind} X$ hold for every F-space of weight $\aleph_1$?

References

[1] Fred Dashiell, *Is Stone-Čech compactification of 0-dimensional space also 0-dimensional?*, `https://mathoverflow.net/questions/93719`. (version: 2018-10-17).

[2] C. H. Dowker, *Local dimension of normal spaces*, Quart. J. Math. Oxford Ser. (2) **6** (1955), 101–120, DOI 10.1093/qmath/6.1.101. MR86286

[3] Ryszard Engelking, *General topology*, 2nd ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989. Translated from the Polish by the author. MR1039321

[4] Leonard Gillman and Meyer Jerison, *Rings of continuous functions*, Graduate Texts in Mathematics, No. 43, Springer-Verlag, New York-Heidelberg, 1976. Reprint of the 1960 edition. MR0407579

[5] Klaas Pieter Hart, *The Čech-Stone compactification of the real line*, Recent progress in general topology (Prague, 1991), North-Holland, Amsterdam, 1992, pp. 317–352, DOI 10.1016/0887-2333(92)90021-I. MR1229130

[6] Klaas Pieter Hart and Jan van Mill, *Covering dimension and finite-to-one maps*, Topology Appl. **158** (2011), no. 18, 2512–2519, DOI 10.1016/j.topol.2011.08.017. MR2847324

[7] F. Hausdorff, *Untersuchungen über Ordnungstypen*, Leipz. Ber. **58** (1906), 106–169.

[8] Felix Hausdorff, *Grundzüge der Mengenlehre*, Leipzig: Verlag von Veit & Comp., 1914 (German). Mit 53 Figuren im Text.

[9] Kenneth Kunen, *Set theory. An introduction to independence proofs*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam-New York, 1980. MR597342

[10] I. K. Lifanov, *The dimension of a product of ordered continua*, Dokl. Akad. Nauk SSSR **177** (1967), 778–781 (Russian). MR0221480

[11] W. McGovern, *Zero-dimensional F-space which is not strongly zero-dimensional*, `https://mathoverflow.net/questions/239324`. (version: 2016-05-20).

[12] Prabir Roy, *Nonequality of dimensions for metric spaces*, Trans. Amer. Math. Soc. **134** (1968), 117–132, DOI 10.2307/1994832. MR227960

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