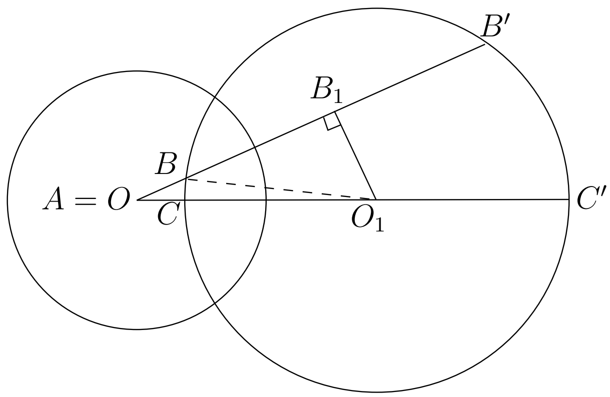
The hyperbolic Pythagorean theorem

To prove some formulas of hyperbolic trigonometry, we begin with the following.

Proposition 1 *Any right triangle $\triangle ABC$ with $\angle C$ being the right angle satisfies*

$$\cos(A) = \frac{\tanh(b)}{\tanh(c)} \quad \text{and} \quad \sin(B) = \frac{\sinh(b)}{\sinh(c)}.$$

Proof: It is your homework to fill in the details in the following proof.



Use the Poincaré disc model and assume that the vertex A is at the center of the disk. (The right angle of $ABC_\triangle$ is at C.) The lines AB and AC are represented by straight lines, the line BC is represented by an arc of a circle centered at O_1. Let B' resp. C' be the second intersection of OB resp OC with this circle and B_1 be the orthogonal projection of O to the line OB.

Using that the Euclidean distance OB equals $\tanh(c/2)$ and that $OB \cdot OB' = 1$ (justify why), prove that the Euclidean distance $BB' = 2/\sinh(c)$. Observe that the Euclidean distance CC' is similarly equal to $2/\sinh(b)$. Due to the Star Trek Lemma, the angle $\angle BO_1B_1$ is equal to $\angle B$. (Why?) Hence

$$\sin(B) = \frac{BB_1}{O_1B} = \frac{BB'}{2O_1C} = \frac{BB'}{CC'} = \frac{\sinh(b)}{\sinh(c)}.$$

Finally, using that $\cos(A) = AB_1/AO_1$, where $AB_1 = OB + BB'/2$ and $AO_1 = AC + CC'/2$, prove that

$$\cos(A) = \frac{\tanh(b)}{\tanh(c)}.$$

$\Diamond$

The hyperbolic Pythagorean theorem is the following statement.

Proposition 2 *Any right triangle $\triangle ABC$ with $\angle C$ being the right angle satisfies* $\cosh(c) = \cosh(a)\cosh(b)$.

Proof: As seen in the previous statement, we have

$$\cos(A) = \frac{\tanh(b)}{\tanh(c)} \quad \text{and} \quad \sin(A) = \frac{\sinh(a)}{\sinh(c)}.$$

Using $\cos^2(A) + \sin^2(A) = 1$ we get

$$1 = \frac{\tanh^2(b)}{\tanh^2(c)} + \frac{\sinh^2(a)}{\sinh^2(c)} = \frac{\tanh^2(b)\cosh^2(c) + \sinh^2(a)}{\sinh^2(c)}.$$

Multiplying both sides by $\sinh^2(c)\cosh^2(b) = (\cosh^2(c) - 1)\cosh^2(b)$ we get

$$(\cosh^2(c) - 1)\cosh^2(b) = \sinh^2(b)\cosh^2(c) + \sinh^2(a)\cosh^2(b).$$

Replacing $\sinh^2(b)$ with $\cosh^2(b) - 1$ and $\sinh^2(a)$ with $\cosh^2(a) - 1$ yields

$$(\cosh^2(c) - 1)\cosh^2(b) = (\cosh^2(b) - 1)\cosh^2(c) + (\cosh^2(a) - 1)\cosh^2(b), \quad \text{that is}$$

$$\cosh^2(c)\cosh^2(b) - \cosh^2(b) = \cosh^2(b)\cosh^2(c) - \cosh^2(c) + \cosh^2(a)\cosh^2(b) - \cosh^2(b).$$

After simplifying and rearranging we obtain

$$\cosh^2(c) = \cosh^2(a)\cosh^2(b).$$

The statement follows from the fact that $\cosh(x) \geq 0$ holds for all real number x. $\Diamond$

Proposition 3 *The previous two statements also imply the following equalities:*

$$\frac{\cos(A)}{\sin(B)} = \cosh(a) \quad \textit{and} \tag{1}$$

$$\cot(A)\cot(B) = \cosh(a)\cosh(b) \tag{2}$$

Proof: Equation (1) Taking the quotient of the the two equations in Proposition 1 we get

$$\frac{\cos(A)}{\sin(B)} = \frac{\tanh(b)}{\tanh(c)} \cdot \frac{\sinh(c)}{\sinh(b)} = \frac{\cosh(c)}{\cosh(b)}.$$

By Proposition 2 we may replace $\cosh(c)$ with $\cosh(a)\cosh(b)$ in the previous equation. Simplifying by $\cosh(b)$ yields (1). In a completely analogous fashion, we also have

$$\frac{\cos(B)}{\sin(A)} = \cosh(b) \tag{3}$$

Multiplying (1) with (3)yields (2). $\Diamond$