Assignment 2

Mandatory questions to be answered orally

1. Give an example of a finite set with a successor operation that satisfies all of Peano's axioms except for Axiom 3. (Axiom 3 states that $x' \neq 1$ for all x.)

2. Give an example of an infinite set with a successor operation that satisfies all of Peano's axioms except for the Axiom of Induction.

3. Give an example of a finite set with a successor relation that satisfies all of Peano's axioms, except for Axiom 2, which has to be replaced by the following weakened axiom: *Each x has at most one successor x'.*

4. Prove that $m + n \neq n$ for every pair of positive integers.

5. Prove that for every positive integer n where $n \neq 1$ there exists exactly one natural number m such that $m' = n$. The number m is called the *predecessor* of n.

6. Prove that if m and n are positive integers then $m + n \neq 1$.

7. Define $2 := 1'$, $3 := 2'$, $4 := 3'$, and $5 := 4'$. Prove that $2 + 3 = 5$.

8. Prove that $n \geq 1$ for every positive integer n.

Mandatory questions to be answered in writing

1. A total order on a set S is *dense* if for every $x, y \in S$ satisfying $x < y$ there is a z satisfying $x < z < y$. Prove that a densely ordered set is not well-ordered.

2. The *symmetric difference* $X \triangle Y$ of two sets consists of all elements of X that do not belong to Y and all elements of Y that do not belong to X. (In other words, $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$, where $X \setminus Y$ is the *set theoretic difference* of X and Y). Define "addition" and "multiplication" on the powerset of a set S by setting

$$X + Y := X \triangle Y \quad \text{and} \quad X \cdot Y := X \cap Y \quad \text{for all } X, Y \subseteq S.$$

   Prove the Distributive Law, for these operations, i.e., prove

$$(X + Y) \cdot Z = X \cdot Z + Y \cdot Z \quad \text{for all } X, Y, Z \subseteq S.$$

3. Prove that multiplication of positive integers satisfies the Cancellation Law, i.e., $xz = yz$ implies $x = y$.

(Turn page for Bonus questions)

Bonus questions

1. Give an example of a set with a successor relation that satisfies all of Peano's axioms except for Axiom 2, which has to be replaced by the following weakened axiom: *Every element has at least one successor (but may have more than one).* (You may want to draw a picture.)

2. Assuming $2 = 1'$ prove that $2 \cdot n = n + n$ for all positive integer n.

3. Give an example of an infinite *well-ordered set* with a successor operation that satisfies all of Peano's axioms except for the Axiom of Induction.