Study Guide for the mandatory part of the final

The actual final exam will have a mandatory and an optional section. The optional questions will be similar to the ones on the previous (sample) tests, and need to be answered only if you do not want me to re-use your average test score. The list of questions below is supposed to help you prepare for the mandatory part of the final. Review also all homework questions assigned after our second test as questions similar to them might appear on the mandatory part of the final.

1. State the contrapositive of the following statement: "If 6 divides x then x is an even number."

2. A machine knows only two operations:

   (A) multiply the number on the screen by 3;

   (B) subtract 1 from the number on the screen.

   At the beginning 1 is displayed on the screen. Prove by smallest counterexample that every positive integer can be made appear on the screen.

3. Prove by induction the following formulas:

   (a) $1 + 2 + \cdots + n = n(n + 1)/2$

   (b) $1^2 + 2^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$

4. Prove that all positive integers satisfy $n < 2^n$.

5. The sequence $a_0, a_1, \ldots, a_n, \ldots$ is given by $a_0 = 7$ and

   $$a_{n+1} = \begin{cases} a_n + 1 & \text{if } a_n \text{ is odd,} \\ a_n/2 & \text{if } a_n \text{ is even.} \end{cases}$$

   Calculate the first seven entries of the sequence.

6. A sequence is given by $a_0 = 3$ and $a_{n+1} = 6 - a_n$. Find an explicit formula for a_n.

7. A sequence is given by $a_0 = 3$ and $a_{n+1} = a_n + 1$. Find an explicit formula for a_n.

8. A sequence is given by $a_0 = 2$, $a_1 = -1$ and $a_n = -a_{n-1} + 6a_{n-2}$. Find an explicit formula for a_n.

9. A sequence is given by $a_0 = 2$, $a_1 = 15$ and $a_n = 10a_{n-1} - 25a_{n-2}$. Find an explicit formula for a_n.

10. Give an example of a relation that is *not* a function.

11. Write the function $f : \{1, 2, 3, 4, 5\} \to \mathbb{Z}$, given by a $f(x) = 2x - 3$ as a set of ordered pairs.

12. Find the number of all functions from a 10 element set into a 5 element set.

13. Which of the following functions is one-to-one? Find the inverse of the one-to-one functions.
    (a) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$,
    (b) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 2x - 4$,
    (c) $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x^2$.
14. A function $f : A \to A$ is one-to-one. Is it also onto? Give a proof if yes, give a counterexample if not.
15. Answer the previous question again, assuming A is a finite set.
16. The functions f and g are given by $f = \{(1,3),(2,4),(3,6)\}$ and $g = \{(2,1),(3,2)(4,3)\}$. Find $f \circ g$ and $g \circ f$.
17. State the pigeonhole principle.
18. Prove that given five distinct lattice point in the plane, at least one of the ten line segments determined by these points has a lattice point as a midpoint.
19. Prove that every sequence of $n^2 + 1$ distinct integers must contain a monotone subsequence of length $n + 1$.
20. Let A be a set. Prove that a function $f : A \to 2^A$ can not be one-to-one and onto at the same time. (Keep in mind that A could be an infinite set.)
21. The permutation $\sigma \in S_5$ is given by $\sigma = (132)(45)$ (cycle notation), the permutation π is given by
    $$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}.$$
    Write $\sigma \circ \pi$ in two row notation and find the cycle notation for $\pi \circ \sigma$.
22. A permutation has an odd number of odd cycles and an even number of even cycles. Is the permutation odd or even?
23. Prove that every permutation may be written as a composition of transpositions.

Good luck. Gábor Hetyei