Zermelo-Fraenkel axioms

1. Axiom of the empty set:

$$\exists x \forall y (y \notin x).$$

There is a set with no elements. (We introduce the symbol $\emptyset$ for it.)

2. Axiom of extensionality:

$$\forall x \forall y \left(x = y \Leftrightarrow \forall z (z \in x \Leftrightarrow z \in y) \right).$$

Two sets are the same if and only if they have the same elements.

3. Axiom of the unordered pair:

$$\forall x \forall y \exists z \left(x \in z \wedge y \in z \wedge \forall u (u \in z \Rightarrow (u = x \wedge u = y)) \right).$$

For any set x and y there is a set z containing x and y as elements and nothing else. (Thus $z = \{x, y\}$.)

4. Axiom of the sum set:

$$\forall x \exists y \left(\forall z (z \in y \Leftrightarrow \exists u (z \in u \wedge u \in x)) \right).$$

The union of a set of sets is a set.

5. Axiom of the power set:

$$\forall x \exists y \forall z \left(z \in y \Leftrightarrow \forall u (u \in z \Rightarrow u \in x) \right).$$

The family of subsets of a set is a set.

6. Axiom of infinity:

$$\exists x \left(\emptyset \in x \wedge \forall y (y \in x \Rightarrow y \cup \{y\} \in x) \right).$$

There is an infinite set.

7. Axiom of subsets (or axiom of comprehension): Let $P(x)$ be a formula with one free (unquantified) variable. Then

$$\forall u \exists v \forall t (t \in v \Leftrightarrow (t \in u \wedge P(t))).$$

Given any set and any proposition $P(x)$, there is a subset of the original set containing precisely those elements x for which $P(x)$ holds.

8. Axiom of replacement: Let $\phi(x, y)$ be a logical formula with two free (unquantified) variables not containing $\forall v$ and $\forall t$. Then

$$\forall x \exists y \phi(x, y) \Rightarrow \forall z \exists u \forall v (v \in u \Leftrightarrow \exists t (t \in z \wedge \phi(t, v))).$$

If Φ is an operation assigning a set to every set then and z is a set then $u = \{\Phi(t) : t \in z\}$ is a set.

9. Axiom of foundation (or axiom of regularity):

$$\forall x (x \neq \emptyset \Rightarrow \exists y (y \in x \wedge \nexists z (z \in y \wedge y \in x))).$$

Every nonempty set x has an element y that is disjoint from x.

10. Axiom of choice:

$$\forall x \left(\emptyset \notin x \Rightarrow \exists y (\forall z (z \in x \Rightarrow \exists ! t (t \in y \wedge t \in z))) \right).$$

Given any set of mutually exclusive non-empty sets, there exists at least one set that contains exactly one element in common with each of the non-empty sets.