Two simplified statements and proofs

The next theorem implies the equivalence of parts (b) and (c) in Theorem 6.6.8, and provides missing proofs of parts of Theorem 4.4.5.

Theorem 1 *For any two nonempty sets A and B, the following two statements are equivalent:*

1. *There is an injective function $f : A \to B$.*
2. *There is a surjective function $g : B \to A$.*

Furthermore, no matter which of the functions f and g is given, we may select the other one in such a way that $g \circ f$ is the identity function of A.

Proof: Suppose there is an injective function $f : A \to B$ and let a be a fixed element of A. For any element b in B define $g(b)$ to be the only element of $f^{-1}(\{b\})$ if b is in the range of f, and set $g(b) = a$ otherwise. Note that $f^{-1}(\{b\})$ has only one element, since f is injective. It is easy to check that $g \circ f$ is the identity function of A. As a consequence, the range of g, which contains the range of $g \circ f$, contains A, so g is surjective.

Suppose next, there is a surjective function $g : B \to A$. For any a in A, the set $g^{-1}(\{a\})$ is not empty. By the Axiom of Choice, we can select an element b_a in each $g^{-1}(\{a\})$ simultaneously. Hence we can define a function $f : A \to B$ by setting $f(a) = b_a$. The function f satisfies that $g \circ f$ is the identity function of A. As a consequence f is one-to-one: if $f(a) = f(a')$ then $g \circ f(a) = g \circ f(a')$ and $a = a'$. $\diamondsuit$

The next theorem is a simplified variant of Theorem 6.6.7. The proof is simplified.

Theorem 2 *A subset X of $\mathbb{N}$ is countable.*

Proof: We attempt to define a bijection $f : \mathbb{N} \to X$ recursively as follows. We set $f(1)$ to be the least element of X. If this does not exist then X is the empty set and it is countable. Remember also that by the *well-ordering principle* every nonempty subset of $\mathbb{N}$ has a least element.

Suppose we have defined $f(1)$, $f(2)$, $\ldots$, $f(m)$. Define $f(m+1)$ to be the least element of $X - \{f(1), f(2), \ldots, f(m)\}$. If there is some m for which this step is not possible then we have a bijection $f : \{1, 2, \ldots, m\} \to X$, and X is finite. Otherwise, we obtain a function $f : \mathbb{N} \to X$. This map is one-to-one, because each element x in the range of f is selected to be the image of some element only once: if m is the least number such that $f(k) = x$ then no smaller i satisfies $f(i) = x$ and also for each $i > k$ the element $f(i)$ is selected from a set that does not contain $f(k)$. It suffices to show the function f is also onto. Assume, by way of contradiction, that some $x \in X$ is not in the range of f. Then $f(x)$ is an element of a set that still contains x. It is also easy to see by induction on m that $f(m) \geq m$ for all m. But then $f(x)$ is at least x and it is the least element of a set containing x, so it is also at most x. We obtain $f(x) = x$, a contradiction. $\Diamond$

Corollary 1 *If $f : A \to B$ is injective and B is countable, then A is countable.*

Note that $A \sim f(A)$ so it suffices to show that $f(A)$ is countable. This is certainly true if B is finite, since the subset of a finite set is finite. If B is countably infinite then there is a bijection $g : B \to \mathbb{N}$. The set $g(f(A))$ has the same cardinality as A and $f(A)$ and it is a subset of $\mathbb{N}$. By the previous theorem it is countable.