Sample Test 2

This study guide is subject to updates until Wednesday November 4.

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The real test will have less questions and you will have about 75 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. The questions below are sample questions related to stating and proving theorems. Besides trying to answer these questions, make sure you also review all homework exercises. The test may also have questions similar to those exercises. During the test, the usage of books or notes, or communicating with other students will not be allowed.

1. Explain why there are exactly n congruence classes modulo n.
2. For which integers $n > 1$ is it true that there are no zerodivisors in $\mathbb{Z}_n$? Justify our answer!
3. Prove that addition and multiplication of congruence classes is associative and commutative in $\mathbb{Z}_n$.
4. Assume p is a prime and a is not a multiple of p. Prove that the congruence $ax \equiv 1 \pmod{p}$ has a solution.
5. Which of the following is a subring of $\mathbb{Z}$: the set of even integers, or the set of odd integers? Justify your answer!
6. Is the Cartesian product of two integral domains an integral domain? Justify your answer!
7. Let R be a ring and let a be any ring element. Prove that the solution x of the equation $a + x = 0_R$ is unique. Explain how this may be used to prove that $a + b = a + c$ implies $b = c$.
8. What is $a \cdot 0_R$ equal to in a ring? Prove your claim!
9. Prove that $-(-a) = a$ in a ring.
10. Prove that $-(a + b) = (-a) + (-b)$ in a ring.
11. Describe the unique solution of the equation $a + x = b$ in a ring.
12. If $ac = bc$ in a ring, does it always follow that $a = b$? When does it follow? Justify your claim with example and/or proof, as appropriate.
13. Prove that every field is an integral domain. Is the converse true?
14. Give an example of a zero divisor and an idempotent element.
15. Let $f : R \to S$ be a ring homomorphism. Prove that $f(0_R) = 0_S$ and that $f(-a) = -f(a)$ for all $a \in R$.

16. Is the map $f : \mathbb{Q} \to \mathbb{Q}$ sending x into $\frac{1}{1+x^2}$ a homomorphism? Justify your answer!

17. Prove that the map $\mathbb{Z} \to \mathbb{Z}_5$ sending each $n \in \mathbb{Z}$ into its congruence class $[n]$ is a surjective homomorphism.

18. (Potential bonus question) Prove that $\mathbb{Z}_6$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$.

19. (Potential bonus question) Prove that $\mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

20. Let R be a ring. When is it true that $\deg(f \cdot g) = \deg(f) + \deg(g)$ holds for all nonzero polynomials $f, g \in R[x]$?

21. Let F be a field. Describe the units of $F[x]$. Justify your description.

Good luck. Gábor Hetyei