Change of basis formulas

Given a basis $\mathcal{B} = \{\mathbf{b_1}, \dots, \mathbf{b_n}\}$ of $\mathbb{R}^n$, we associate to it the matrix $P_{\mathcal{B}} = [\mathbf{b_1}, \dots, \mathbf{b_n}]$.

Example. When $n = 2$ set $\mathbf{b_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{b_2} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$. We then have

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \right\} \quad \text{and} \quad P_{\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ 1 & 1/2 \end{bmatrix}.$$

The $\mathcal{B}$*-coordinates* of a vector $\mathbf{x}$ are its coefficients in the basis $\mathcal{B}$. They are recorded as the vector $[\mathbf{x}]_{\mathcal{B}}$. When we do not indicate the basis, then we have the standard basis $\{\mathbf{e_1}, \dots, \mathbf{e_n}\}$ in mind. The coordinates in the standard basis are given by the equation

$$[\mathbf{x}] = P_{\mathcal{B}} \cdot [\mathbf{x}]_{\mathcal{B}}. \tag{1}$$

Example. If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $P_{\mathcal{B}}$ is the matrix above, then $[\mathbf{x}] = \begin{bmatrix} 2 & 0 \\ 1 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

If you are given $[\mathbf{x}]$ and you are looking for $[\mathbf{x}]_{\mathcal{B}}$ then (1) implies

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \cdot [\mathbf{x}]. \tag{2}$$

Example. For the matrix $P_{\mathcal{B}}$ as above we have $P_{\mathcal{B}}^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1 & 2 \end{bmatrix}$. Hence for the vector $[\mathbf{x}] = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, we get

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1/2 & 0 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

If you are given a second basis $\mathcal{C}$ and a vector $[\mathbf{x}]_{\mathcal{B}}$ given in the basis $\mathcal{B}$, and you want to find $[\mathbf{x}]_{\mathcal{C}}$, you may do so by combining equations (1) and (2) as follows.

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1} \cdot [\mathbf{x}] = P_{\mathcal{C}}^{-1} \cdot P_{\mathcal{B}} \cdot [\mathbf{x}]_{\mathcal{B}}. \tag{3}$$

The matrix $P_{\mathcal{C}}^{-1} \cdot P_{\mathcal{B}}$ is denoted by $P_{\mathcal{C} \leftarrow \mathcal{B}}$ in our textbook.

Example. For the matrix

$$P_{\mathcal{C}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

we have

$$P_{\mathcal{C}}^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \quad \text{and} \quad P_{\mathcal{C} \leftarrow \mathcal{B}} = P_{\mathcal{C}}^{-1} \cdot P_{\mathcal{B}} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 1 & 1/2 \end{bmatrix} = \begin{bmatrix} 3/2 & 1/4 \\ -1/2 & 1/4 \end{bmatrix}.$$

For the matrix $[\mathbf{x}]_B$ as above, we get

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 3/2 & 1/4 \\ -1/2 & 1/4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

Note that, since we know $[\mathbf{x}]$ in this example, we may also find $[\mathbf{x}]_\mathcal{C}$ using (2) as follows:

$$[\mathbf{x}]_\mathcal{C} = P_\mathcal{C}^{-1} \cdot [\mathbf{x}] = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

The matrix $[T]$ of a linear transformation T is the matrix, whose i-th column is $T(\mathbf{e}_i)$.

Example. The matrix of the rotation around the origin by positive 90 degree is $[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Given a vector $[\mathbf{x}]$, the coordinates of $T(\mathbf{x})$ are given by

$$[T(\mathbf{x})] = [T] \cdot [\mathbf{x}]. \tag{4}$$

Example. The rotation above takes our sample vector $[\mathbf{x}]$ into

$$[T(\mathbf{x})] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}.$$

The effect of the transformation T in a different basis $\mathcal{B}$ can be expressed using (1) and (2) as follows:

$$[T(\mathbf{x})]_\mathcal{B} = P_\mathcal{B}^{-1} \cdot [T(\mathbf{x})] = P_\mathcal{B}^{-1} \cdot [T] \cdot [\mathbf{x}] = P_\mathcal{B}^{-1} \cdot [T] \cdot P_\mathcal{B} \cdot [\mathbf{x}]_\mathcal{B}.$$

Hence the matrix of the transformation T in the basis $\mathcal{B}$ is

$$[T]_\mathcal{B} = P_\mathcal{B}^{-1} \cdot [T] \cdot P_\mathcal{B}. \tag{5}$$

Example. The matrix of the rotation around the origin by positive 90 degree in our basis $\mathcal{B}$ is

$$\begin{bmatrix} 1/2 & 0 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 1 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 1 & 1/2 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/4 \\ 5 & 1/2 \end{bmatrix}$$

Rotating around the origin by positive 90 degrees our vector with $\mathcal{B}$-coordinates $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ gives

$$[T(\mathbf{x})]_\mathcal{B} = \begin{bmatrix} -1/2 & -1/4 \\ 5 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 16 \end{bmatrix}.$$

The standard coordinates of this vector are

$$[T(\mathbf{x})] = \begin{bmatrix} 2 & 0 \\ 1 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 16 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}.$$