Assignment 11

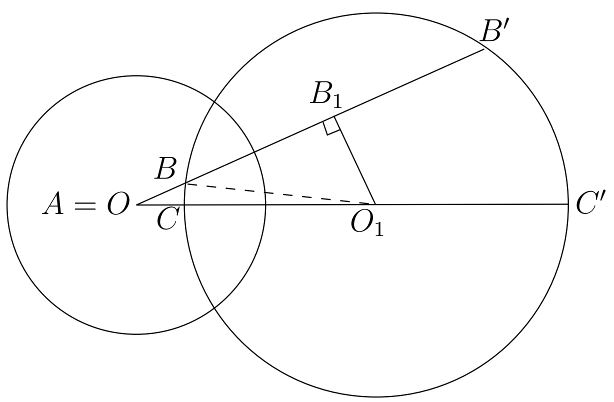
Oral question

1. Prove that the distance function $d(A,B) = |\log(AB, PQ)|$ of the Poincaré disk model is additive: if $A * C * B$ on a Poincaré line then $d(AC) + d(CB) = d(AB)$. Fix a Poincaré line with ideal points P and Q and a point A on it. Move another point B along the Poincaré line from P to Q. Show that $d(A,B)$ changes from ∞ to 0 and then back to ∞.

Question to be answered in writing

1. A hyperbolic circle centered at C of radius r is the set of all points A satisfying $d(A,C) = r$. Prove that a hyperbolic circle corresponds to a circle in the Poincaré disk model, although its center may not be equal to its Euclidean center. (Hint: prove the statement in the case when $C = P$ first, where P is the center of the Poincaré disk. Use then the fact that reflection about the perpendicular bisector of PC takes a hyperbolic circle centered at P into a hyperbolic circle centered at C, and that this reflection corresponds to an inversion about a circle.)

2. Complete the following proof of the *hyperbolic Pythagorean theorem* (Theorem 16.1) which states the following: *Any right triangle $\triangle ABC$ with $\angle C$ being the right angle satisfies* $\cos(A) = \tanh(b)/\tanh(c)$.



Use the Poincaré disc model and assume that the vertex A is at the center of the disk. (The right angle of $ABC_\triangle$ is at C.) The lines AB and AC are represented by straight lines, the line BC is represented by an arc of a circle centered at O_1. Let B' resp. C' be the second intersection of OB resp OC with this circle and B_1 be the orthogonal projection of O to the line OB.

Using that the Euclidean distance OB equals $\tanh(c/2)$ and that $OB \cdot OB' = 1$ (justify why), prove that the Euclidean distance $BB' = 2/\sinh(c)$. Observe that the Euclidean distance CC' is similarly equal to $2/\sinh(b)$. Due to the Star Trek Lemma, the angle $\angle BO_1B_1$ is equal to $\angle B$. (Why?) Hence

$$\sin(B) = \frac{BB_1}{O_1B} = \frac{BB'}{2O_1C} = \frac{BB'}{CC'} = \frac{\sinh(b)}{\sinh(c)}.$$

Finally, using that $\cos(A) = AB_1/AO_1$, where $AB_1 = OB + BB'/2$ and $AO_1 = AC + CC'/2$, prove that

$$\cos(A) = \frac{\tanh(b)}{\tanh(c)}.$$