A replacement for Theorem 26.5

Theorem 26.5 in our textbook uses the Fundamental Theorem of Calculus [Theorem 34.3] that is covered only later. To avoid this apparent contradiction, we prefer to derive it from the following theorem.

Theorem Assume that a sequence of functions (f_n) converges pointwise to the function f on an interval $[a,b]$. Assume furthermore that each f_n is differentiable on $[a,b]$ and that $f_n \to g$ uniformly on $[a,b]$. Then f is differentiable on $[a,b]$ and we have $f' = g$.

Proof: Consider any $x_0 \in [a,b]$, and let us fix $\varepsilon > 0$. By the Mean Value Theorem (applied to $f_n - f_m$), for all $x \in [a,b]$ and all pairs of positive integers (m,n), there is a z between x and x_0 such that

$$(f_m(x) - f_n(x)) - (f_m(x_0) - f_n(x_0)) = (x - x_0)(f'_m(z) - f'_n(z)).$$

Hence for $x \neq x_0$ we have

$$\left| \frac{f_m(x) - f_m(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} \right| \leq \sup_{z \in [a,b]} |f'_m(z) - f'_n(z)|.$$

Since the sequence (f'_n) is uniformly Cauchy, there is a N_1 such that $\sup_{z\in[a,b]} |f'_m(z) - f'_n(z)| < \varepsilon$ whenever $m, n > N_1$. Thus we may write

$$\left| \frac{f_m(x) - f_m(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} \right| < \varepsilon \quad \text{for } m, n > N_1.$$

Let us keep n fixed and let $m \to \infty$. Then we get

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} \right| \leq \varepsilon \quad \text{for } n > N_1. \tag{1}$$

Since $g(x_0) = \lim_{n\to\infty} f'_n(x_0)$, there is an N_2 such that

$$|f'_n(x_0) - g(x_0)| < \varepsilon \quad \text{for } n > N_2. \tag{2}$$

Finally, by the definition of the derivative, there is a $\delta > 0$ such that

$$\left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - f'_n(x_0) \right| < \varepsilon \quad \text{for } |x - x_0| < \delta. \tag{3}$$

Combining equations (1), (2), and (3) for any n satisfying $n > \max(N_1, N_2)$ we get that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \right| < 3\varepsilon \quad \text{for } |x - x_0| < \delta.$$

The argument may be repeated for any $\varepsilon > 0$ showing that $f'(x_0) = g(x_0)$.