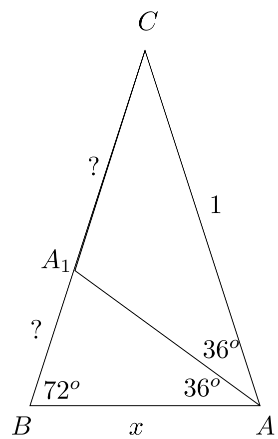
Assignment 7

No oral questions are assigned due to the upcoming midterm.

Questions to be answered in writing

1. Use the picture below to find an exact formula for $\cos(72^o) = x/2$.

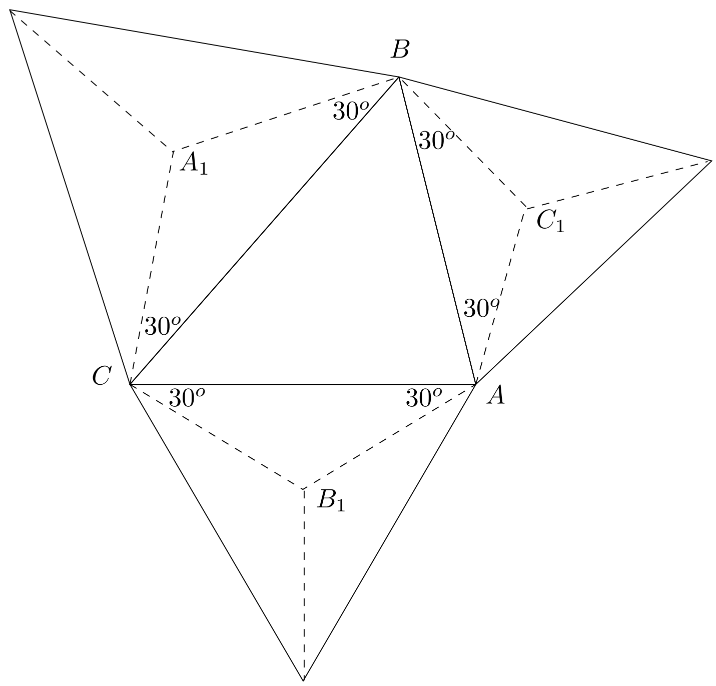


   Prove your claim using similarity of triangles and the angle bisector theorem. Explain how your formula may be used to construct a regular pentagon if:

   (a) the length of one side is given;
   (b) the radius of the circumscribed circle is given.

2. Let a and b the side lengths of a parallelogram, and c and d the lengths of its diagonals. Prove that $2(a^2 + b^2) = c^2 + d^2$. (In other words, the sum of the lengths of the squares of the diagonals equals the sum of the squares of the side lengths.)

3. Prove Napoleon's theorem: Given an arbitrary triangle $ABC_\triangle$, the centers of the equilateral triangles exterior to $ABC_\triangle$ form an equilateral triangle. (Illustration and hints on next page.)



Hints: Represent the points A, B, C, A_1, B_1, C_1 with complex numbers a, b, c, a_1, b_1, c_1. Observe that multiplying with

$$\rho := \frac{1}{\sqrt{3}}\left(\cos(30^o) + i \cdot \sin(30^o)\right)$$

rotates the vector $\overrightarrow{BA} = a - b$ into $\overrightarrow{BC_1} = c_1 - b$. Use this observation to express c_1 in terms of a, b and ρ. Express then a_1 and c_1 similarly in terms of a, b, c and ρ. Show that $c_1 - a_1$ is obtained by multiplying $b_1 - a_1$ with

$$\frac{\rho}{1-\rho} = \frac{2\rho - 1}{\rho} = \frac{\rho - 1}{2\rho - 1}.$$

It is probably easier to do so if you find the quadratic equation whose roots are ρ and its conjugate. Finally show that

$$\frac{\rho}{1-\rho} = \cos(60^o) + i \cdot \sin(60^o)$$

meaning that $\overrightarrow{A_1C_1}$ is obtained from $\overrightarrow{A_1B_1}$ by a 60^o rotation.