Fibonacci-type sequences

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A Fibonacci-type sequence $a_0, a_1, \ldots$ is given by the recursion formula $a_{n+2} + ba_{n+1} + ca_n = 0$ and by the initial values for a_0 and a_1. For example, for the Fibonacci numbers: $b = c = -1$ and $a_0 = a_1 = 1$. Introducing $F(x) = \sum_{n=0}^{\infty} a_n x^n$ we get

$$\begin{aligned}(1+bx+cx^2)F(x) &= (1+bx+cx^2)\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n + b\sum_{n=0}^{\infty} a_n x^{n+1} + c\sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=2}^{\infty} \underbrace{(a_n + ba_{n-1} + ca_{n-2})}_{0} x^n + (a_0 + a_1 x) + ba_0 x = a_0 + (a_1 + ba_0)x,\end{aligned}$$

and so

$$F(x) = \frac{a_0 + (a_1 + ba_0)x}{1 + bx + cx^2}. \tag{1}$$

For example, for the Fibonacci numbers, we obtain the generating function $\frac{1}{1-x-x^2}$. If possible, we want to rewrite $1 + bx + cx^2$ as $(1 - r_1 x)(1 - r_2 x)$. This is possible if the *characteristic equation*

$$x^2 + bx + c = 0$$

has two distinct nonzero roots. (Note: if any of the roots is 0 then $c = 0$, so $1 + bx + cx^2 = 1 + bx$, and we are already in good shape). The reason for this quest is that one can always solve

$$\frac{1}{(1-r_1 x)(1-r_2 x)} = \frac{A_1}{1-r_1 x} + \frac{A_2}{1-r_2 x}. \tag{2}$$

and thus rewrite $F(x)$ into a simpler form. In fact, (2) is equivalent to

$$1 = A_1(1 - r_2 x) + A_2(1 - r_1)x = A_1 + A_2 - (A_1 r_2 + A_2 r_1)x$$

So we need to solve the system

$$\begin{aligned} A_1 + A_2 &= 1 \\ A_1 r_2 + A_2 r_1 &= 0 \end{aligned}$$

for the unknowns A_1 and A_2. Form the first equation we may express A_2 as $A_2 = 1 - A_1$, and so in the second equation we get $A_1 r_2 + (1 - A_1) r_1 = 0$, that is $A_1(r_2 - r_1) = -r_1$. From here

$$A_1 = \frac{-r_1}{r_2 - r_1} \quad \text{and} \quad A_2 = \frac{r_2}{r_2 - r_1}. \tag{3}$$

Once we have rewritten our generating function using (2) we may use the formula:

$$\frac{1}{(1 - rx)} = \sum_{n=0}^{\infty} r^n x^n.$$

Note that

$$\frac{x}{(1 - rx)} = \sum_{n=0}^{\infty} r^n x^{n+1} = \sum_{n=1}^{\infty} r^{n-1} x^n.$$

The only situation when this method does not work is when the characteristic equation $x^2 + bx + c = 0$ has a double root r. In that case we will need to use

$$\begin{aligned} \frac{1}{(1 - rx)^2} = (1 - rx)^{-2} &= \sum_{n=0}^{\infty} \binom{-2}{n} (-1)^n r^n x^n = \sum_{n=0}^{\infty} \left(\!\binom{n}{2}\!\right) r^n x^n \\ &= \sum_{n=0}^{\infty} \binom{n + 2 - 1}{n} r^n x^n = \sum_{n=0}^{\infty} (n + 1) r^n x^n. \end{aligned}$$

For the Fibonacci numbers themselves, the characteristic equation is $x^2 - x - 1 = 0$ which has two distinct roots $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$. Note also that $r_2 - r_1 = -\sqrt{5}$, so (2) and (3) yields

$$\begin{aligned} F(x) &= \frac{1}{1 - x - x^2} = \frac{1 + \sqrt{5}}{2\sqrt{5}} \frac{1}{1 - \frac{1+\sqrt{5}}{2} x} + \frac{\sqrt{5} - 1}{2\sqrt{5}} \frac{1}{1 - \frac{1-\sqrt{5}}{2} x} \\ &= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} x^n - \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} x^n \end{aligned}$$

Therefore

$$a_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} \right)$$

An example for the double root situation would be $a_{n+2} - 6a_{n+1} + 9a_n = 0$ where the characteristic equation is $x^2 - 6x + 9 = 0$, with double root $r = 3$. Assuming $a_0 = 0$ and $a_1 = 1$, from (1) we get

$$F(x) = \frac{x}{1 - 6x + 9x^2} = \frac{x}{(1 - 3x)^2} = \sum_{n=0}^{\infty} (n + 1) 3^n x^{n+1} = \sum_{n=1}^{\infty} n 3^{n-1} x^n,$$

and so $a_n = n3^{n-1}$ for $n \geq 1$.