Study Guide for Test 2.

This document may undergo substantial changes until our class on March 18. After that, only minor corrections may be possible

The real test will have less questions and you will have about 80 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. Below you find sample questions and indications which theorems and proofs you will have to remember from the book. Review also all homework questions assigned after our first test as questions similar to them might appear on the test.

1. Define the inverse of a square matrix.
2. Let T be an invertible linear transformation between finite dimensional vector spaces. Prove that the domain and the range of T must have the same dimension.
3. Prove that two finite dimensional vector spaces are isomorphic if and only if they have the same dimension.
4. Describe how the coordinate matrix changes if you change the ordered basis. Outline the proof of your formula.
5. Prove that the trace of similar square matrices is equal.
6. Prove that any vector space V may be embedded into the dual of its dual. Under what conditions is this embedding an isomorphism?
7. Let V be the vector space of all polynomials of degree at most n. For $i = 0, 1, \ldots, n$ and any polynomial $p(x)$, define $f_i(p(x))$ by $f_i(p(x)) := p^{(i)}(0)$. Here $p^{(i)}(x)$ is the i-th derivative of p (in particular, $p^{(0)}(x) = p(x)$. Prove that the set linear functionals $\{f_0, \ldots, f_n\}$ is a basis of V^*, and find a basis of V for which it is the dual basis.
8. Explain why elementary row and column operations are rank preserving. (No need to prove Theorem 3.4.)
9. Explain what algorithm to use to transform a matrix into the form given in Theorem 3.6 (might give specific example to work out for test).
10. Explain why the row rank is equal to the column rank.
11. Explain why each invertible matrix is a product of elementary matrices, and how the algorithm to find such a decomposition may be used to invert a matrix. (Hint: be prepared to find the inverse of any elementary matrix).
12. Describe the general solution of a system of linear equations in terms of a particular solution and the nullspace of the coefficient matrix. Prove your statement.

13. State and prove a necessary and sufficient condition of the solvability of the system of linear equations $Ax = b$ in terms of the rank of A and the augmented matrix $(A|b)$.
14. Let A and B be 2×2 matrices. Prove that $\det(AB) = \det(A)\det(B)$.
15. State and prove the elementary product expansion formula of the determinant.
16. Prove that the determinant may be expanded by any row or any column. (State and prove the appropriate cofactor expansion formula).
17. Prove that transposing a matrix leaves its determinant unchanged.
18. How can you express the determinant of an upper diagonal matrix in terms of its main diagonal entries? Prove your statement.
19. State the Cauchy-Binet formula and illustrate it by evaluating the determinant of AB where A is a 2×3 matrix and B is a 3×2 matrix.
20. State and prove Cramer's formula.
21. Explain how the notion of the determinant is related to the notion of n-dimensional volume. (No proofs necessary.)
22. Express the inverse of a matrix in terms of the determinant and a matrix whose entries are cofactors of A. Prove your statement for 3×3 matrices.

Good luck. Gábor Hetyei