Study Guide for Test 2.

The real test will have less questions and you will have 75 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. Below you find sample questions and indications which theorems and proofs you will have to remember from the book. Review also all past homework questions as questions similar to them might appear on the test.

1. Which of 2 and $\{2\}$ is an element of $\{1, 2, 3\}$ and which is a subset?
2. If A has 10 elements, how many elements are there in the powerset of A?
3. Which of the following is true for integers $\forall x \exists y(x = y)$ or $\exists x \forall y(x = y)$? Prove the true statement, give a counterexample to the false one.
4. Negate the sentence $\forall x \exists y \exists z(xy = z)$.
5. If $\forall x \exists y P(x, y)$ is true, can we conclude that $\exists x \exists y P(x, y)$ is also true? Why?
6. How do you define the equality of two sets, and why does this definition imply that there is only one empty set?
7. Let $A = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}(x = y^2)\}$ and let $B = \{x \in \mathbb{Z} : x \geq 0\}$. Which set is a subset of the other? Prove your statement.
8. Prove that $|A \cup B| = |A| + |B| - |A \cap B|$ and explain how this implies the addition principle.
9. Find the number of multiples of 4 or 14 between 1 and $10,000$.
10. Let $A = \{x \in \mathbb{Z} : 4|x\}$ and $B = \{x \in \mathbb{Z} : 6|x\}$. Describe the symmetric difference $A \triangle B$.
11. Either prove or give a counterexample to the statement: "If a relation is not reflexive then it is irreflexive".
12. Either prove or give a counterexample to the statement: "If a relation is irreflexive then it is not reflexive".
13. Similarly to the previous two questions, explain the difference between "not symmetric" and "antisymmetric" relations. Which property implies the other one, and which does not?
14. Determine whether each of the relations below is reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.
    (a) The relation $x|y$ ("x divides y") on the set of positive integers.
    (b) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y - x > 1\}$.
    (c) $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (2, 1), (3, 2)\}$.

15. Let R be the relation on $\mathbb{Z} \times \mathbb{Z}$ defined by $(x, y) \in R$ if x and y have the same parity. Prove that R is an equivalence relation. How many equivalence classes are there?
16. Explain how an equivalence relation defines a partition and vice versa.
17. State and prove the multinomial theorem.
18. Find the coefficient of $x^3y^2z^2$ in $(x + y + z)^7$.
19. How many ways are there to line up 5 apples, 3 oranges, and 2 peaches on a shelf? Explain your answer.
20. Explain the meaning of the symbol $\binom{n}{k}$ in terms of selecting subsets from a set and prove that $\binom{n}{k} = \frac{(n)_k}{k!}$.
21. Explain why the previous formula implies $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
22. State and prove the binomial theorem.
23. Express $\sum_{k=1}^{n-1} k$ as a binomial coefficient.
24. State and prove Pascal's identity.
25. Explain why each row in Pascal's triangle is symmetric. (=If you read it "backwards", you get the same list.)
26. Explain how $\binom{x}{k}$ can be defined for any natural number k and variable x. Using this definition, calculate $\binom{2/3}{3}$.
27. Explain why $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.
28. How many ways are there to select 5 coins from an unlimited supply of nickels, dimes, and quarters?
29. Express $\left(\!\binom{n}{k}\!\right)$ as a binomial coefficient and prove your claim.
30. State the inclusion-exclusion formula for four sets A, B, C, D. Explain why using Venn-diagrams may be misleading.
31. State the inclusion-exclusion formula for n sets.
32. How many ternary lists of length 5 have n repeated consecutive digits? Give a formula using inclusion-exclusion. Is there a simpler way to answer the question?
33. Five people go to a party, each wearing a hat. Upon arrival, they leave the hats in the wardrobe. When they leave, they pick up a hat as random. What is the probability no one picked his or her own hat?

Good luck. Gábor Hetyei