Sample Final Exam Questions (Mandatory Part)

The actual final exam will have a mandatory and an optional section. The optional questions will be similar to the ones on the previous (sample) tests, and need to be answered only if you do not want me to re-use your average (unadjusted) test score. The list of questions below is supposed to help you prepare for the mandatory part of the final. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. Below I am only listing questions related to the definitions, theorems, and proofs I expect you to know. There will be also application exercises, similar to the already discussed homework questions.

1. What is the coefficient of x^2yz^3 in $(x+y+z)^6$?
2. What is the number of partitions of an n-set such that there are exactly p_j blocks of size j? Justify your answer!
3. Express $(-1)^n\binom{1/2}{n}$ as a constant multiple of a Catalan number.
4. The Catalan number C_n is defined as the number of sequences $a_1,\ldots,a_{2n}$ such that exactly n of the a_is is 1, the remaining a_is are -1, and we have $a_1+a_2+\cdots+a_m\geq 0$ for all $n\leq n$. Express C_n using binomial coefficients. Prove your formula. (It is your choice whether you want to use the reflection principle, generating functions, or some other method.)
5. Prove that the number of ways to climb a stairway of n steps by taking 1 or 2 steps at a time is the Fibonacci number F_n. Use this observation to express F_n as a sum of binomial coefficients of the form $\binom{n-i}{i}$. Prove your formula.
6. Give a closed-form formula for the Fibonacci number F_n and prove it.
7. Use the closed-form formula for F_n to show that, for large n, the quotient F_{n+1}/F_n approximately equals the golden ratio $\frac{1+\sqrt{5}}{2}$.
8. Prove by strong induction that the Lucas number L_n is given by $L_n=F_{n-2}+F_n$. Explain why this formula shows that L_n counts the tilings of the circular n-board with 1- and 2-tiles.
9. Find the ordinary generating function $f_k(x)=\sum_{n=0}^{\infty}S(n,k)x^n$ of the Stirling numbers of the second kind $S(n,k)$. Prove your formula.
10. Write x^4-2x as a linear combination of the polynomials $(x)_4$, $(x)_3$, $(x)_2$, $(x)_1$ and $(x)_0$.
11. Write $(x)_4-2(x)_2$ as a linear combination of the powers of x.
12. Prove the formula
$$\Delta^m f(n)=\sum_{k=0}^{m}(-1)^k\binom{m}{k}f(n+m-k).$$

13. Prove the formula
$$f(n)=\sum_{k=0}^{n}\binom{n}{k}\Delta^k f(0)\quad\text{for } n\geq 0,$$
and explain how this formula may be used to find a closed form formula for a higher order arithmetic sequence.

14. Find a closed-form formula for $f(n)=1^3+2^3+\cdots+n^3$.

15. Prove that the Stirling number of the second kind $s(n,k)$ is given by $s(n,k)=(-1)^{n-k}c(n,k)$ where $c(n,k)$ is the number of permutations of $\{1,2,\ldots,n\}$ with k cycles.

16. State the definition of a partially ordered set and give an example of an infinite poset that is locally finite.

17. Draw the Hasse diagram of the partially ordered set D_{20} of positive divisors of 20, ordered by the relation "divides".

18. Define the incidence algebra and the zeta function of a locally finite poset. Explain how the convolution operation corresponds to matrix multiplication and the relation between the matrices representing the zeta function and the Möbius function.

19. Consider the set of finite subsets of a set, partially ordered by inclusion. Given $X\subseteq Y$, what is the value $\mu(X,Y)$ of the Möbius function μ, evaluated at (X,Y)? Justify your formula!

20. State and outline the proof of the Möbius inversion formula.

21. What is the consequence of the Möbius inversion formula for the set $\mathbb{N}$ of natural numbers, totally ordered by the usual $\leq$ relation?

22. Consider the set D_n of positive divisors of n, partially ordered by the relation "divides". Given $x,y\in D_n$ what is the value $\mu(x,y)$ of the Möbius function μ, evaluated at (X,Y)? Justify your formula! (Note: in your justification you may need to use the formula for the Möbius function of a product of posets. You do not need to know how to prove this formula, but you have to know it to be able to use it.)

Good luck. Gábor Hetyei