Inclusion-exclusion formulas

Let A_1, A_2, …, and A_n be subsets of the same finite set X. The *inclusion-exclusion formulas* allow us to count the elements in the union $\bigcup_{i=1}^n A_i$ and in its complement.

Theorem 1 *We have*

$$\left|X \setminus \bigcup_{i=1}^n A_i\right| = |X| - \sum_{i=1}^n |A_i| + \sum_{1\leq i<j\leq n} |A_i \cap A_j| - \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots A_n|, \quad \textit{that is,}$$

$$\left|X \setminus \bigcup_{i=1}^n A_i\right| = \sum_{r=0}^n (-1)^r \sum_{1\leq i_1<i_2<\cdots<i_r\leq n} |A_{i_1} \cap A_{i_2} \cap \cdots A_{i_r}|.$$

Proof: Let x be any element of X. Assume x belongs to exactly j sets from $A_1, \ldots A_n$. (Here j is any integer between 0 and n.) Without loss of generality we may assume that x belongs to A_1, A_2, …, A_j, but does not belong to A_{j+1}, A_{j+2}, …, A_n. This element is counted zero times on the left hand side if $j > 0$, and it is counted once if $j = 0$. It suffices to show that x is counted the same number of times on the right hand side. Clearly x belongs to the intersection $A_{i_1} \cap A_{i_2} \cap \cdots A_{i_r}$ if and only if the index set $\{i_1, \ldots, i_r\}$ is a subset of $\{1, 2, \ldots, j\}$, and there are $\binom{j}{r}$ such subsets. Thus x is counted with multiplicity $\sum_{r=0}^j (-1)^r \binom{j}{r}$ on the right hand side. By the binomial theorem, we have

$$\sum_{r=0}^j (-1)^r \binom{j}{r} = (1-1)^j = \delta_{0,j}$$

where $\delta_{0,j}$ is the Kronecker delta. Therefore x is counted the same number of times on both sides. $\Diamond$

Subtracting both sides of the equation stated in the Theorem above from $|X|$, we obtain an important variant of the inclusion-exclusion formula:

$$\left|\bigcup_{i=1}^n A_i\right| = \sum_{i=1}^n |A_i| - \sum_{1\leq i<j\leq n} |A_i \cap A_j| + \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots A_n|, \quad \text{that is,}$$

$$\left|\bigcup_{i=1}^n A_i\right| = \sum_{r=1}^n (-1)^{r-1} \sum_{1\leq i_1<i_2<\cdots<i_r\leq r} |A_{i_1} \cap A_{i_2} \cap \cdots A_{i_r}|.$$