

## Sample Final Exam Questions (Mandatory Part)

The actual final exam will have a mandatory and an optional section. The optional questions will be similar to the ones on the previous (sample) tests, and need to be answered only if you do not want me to re-use your average (adjusted) test score. The list of questions below is supposed to help you prepare for the mandatory part of the final. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. Below I am only listing questions related to the definitions, theorems, and proofs I expect you to know. There will be also application exercises, similar to the already discussed homework questions.

1. Prove by strong induction that the Lucas number  $L_n$  is given by  $L_n = F_{n-2} + F_n$ . Explain why this formula shows that  $L_n$  counts the tilings of the circular  $n$ -board with 1- and 2-tiles.
2. Find the ordinary generating function  $f_k(x) = \sum_{n=0}^{\infty} S(n, k)x^n$  of the Stirling numbers of the second kind  $S(n, k)$ . Prove your formula.
3. Let  $g_k(x)$  be the exponential generating function of the Stirling numbers of the second kind  $S(n, k)$ . Write an ordinary differential equation for  $g_k(x)$  that allows to compute it from  $g_{k-1}(x)$ .
4. Prove that the Bell number  $B_n$  satisfies

$$B_n = \frac{1}{e} \sum_{j=0}^{\infty} \frac{j^n}{j!}.$$

5. Write  $x^4 - 2x$  as a linear combination of the polynomials  $(x)_4, (x)_3, (x)_2, (x)_1$  and  $(x)_0$ .
6. Write  $(x)_4 - 2(x)_2$  as a linear combination of the powers of  $x$ .
7. Prove the formula

$$\Delta^m f(n) = \sum_{k=0}^m (-1)^k \binom{m}{k} f(n + m - k).$$

8. Prove the formula

$$f(n) = \sum_{k=0}^n \binom{n}{k} \Delta^k f(0) \quad \text{for } n \geq 0,$$

and explain how this formula may be used to find a closed form formula for a higher order arithmetic sequence.

9. Find a closed-form formula for  $f(n) = 1^3 + 2^3 + \dots + n^3$ .
10. Prove that the Stirling number of the first kind  $s(n, k)$  is given by  $s(n, k) = (-1)^{n-k} c(n, k)$  where  $c(n, k)$  is the number of permutations of  $\{1, 2, \dots, n\}$  with  $k$  cycles.
11. Prove that the number of integer partitions of  $n$  into  $k$  parts is the same as the number of integer partitions of  $n$  with largest part of size  $k$ .

12. State and prove a formula for  $P(n, 2)$ .
13. Prove that  $P(n, 3) = \left\{ \frac{n^2}{12} \right\}$ . Here  $\{x\}$  is the nearest integer to  $x$ .
14. Is the set of all integers a partially ordered set, with respect to the relation “divides”? How about the set of positive integers? Justify your answer!
15. Give an example of a partially ordered set that is not a lattice.
16. Prove that the partially ordered set  $D_n$  of positive divisors of  $n$  is a lattice, by describing the join and the meet of two positive integers.
17. Draw the Hasse diagram of the partially ordered set  $D_{20}$  of positive divisors of 20, ordered by the relation “divides”.
18. Consider the set of subsets of  $\{1, 2, \dots, n\}$ , partially ordered by inclusion. What is the width of this set?
19. State and prove Sperner’s theorem.

Good luck.

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