Name: **Student ID:**

Sample Test I.

The actual test will have less questions and perhaps one bonus question. You will have 75 minutes to answer them, without using your notes or communicating with other students. You will have to give the simplest possible answer and show all your work. Besides trying to answer the sample questions listed below, make sure you also review all homework questions that we already discussed in class as any of them may appear on the test.

1. Prove by induction that
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)\cdot n}=\frac{n-1}{n}.$$
holds for every positive integer $n \geq 2$.

2. State and prove Pascal's identity.

3. Evaluate the following sums using the binomial theorem.

   (a) $\displaystyle\sum_{k=0}^{n}\binom{n}{k}(n-1)^k$.

   (b) $\displaystyle\sum_{k=1}^{n}\binom{n}{k}\cdot k\cdot 3^{k-1}$. (Hint: Take the derivative of the appropriate polynomial.)

4. Prove that, for any triangular number n, the number $8n+1$ is a perfect square.

5. Prove that the relation "divides" is a partial order on positive integers.

6. Prove the correct statement, provide a counterexample to the false one:

   (a) If $d|a$ and $d|b$ then $d|a+b$.

   (b) If $d|a+b$ then $d|a$ and $d|b$.

7. In which sense is the greatest common divisor of two integers the "greatest"?

8. Explain why being able to write $\gcd(a,b)$ in the form $ax+by$ implies that every common divisor of a and b divides $\gcd(a,b)$.

9. Using Euclid's Algorithm, find the greatest common divisor of 540 and 246 and write it in the form $540x+246y$. (Note: You will not get full credit unless you record the individual steps of the algorithm.)

10. Prove that Euclid's algorithm finds the greatest common divisor of two integers.

11. What is the relation between the greatest common divisor $\gcd(a,b)$ and the least common multiple $\operatorname{lcm}(a,b)$ of the integers a and b? Prove your claim.

12. State a necessary and sufficient condition, in terms of $\gcd(a, b)$ and c, for the Diophantine equation $ax + by = c$ to have a solution. Prove your claim.
13. Solve the Diophantine equation $6x - 21y = 15$.
14. State and prove Euclid's lemma.
15. Which principle of induction do you have to use to prove the existence part of the fundamental theorem of arithmetic? Explain why.
16. Explain how to use Euclid's lemma in the proof of the fundamental theorem of arithmetic.
17. Prove that $\sqrt{2}$ is irrational.
18. Prove that there are infinitely many primes.
19. Prove that the nth prime number p_n is at most $2^{2^{n-1}}$.
20. Prove that congruence is an equivalence relation.
21. Prove that congruence is compatible with addition, subtraction, and multiplication.
22. Prove that every integer has a unique base b representation. Here b is any positive integer that is larger than 1.
23. Let $p(x)$ be any polynomial with integer coefficients. Prove that $a \equiv b$ modulo n implies $p(a) \equiv p(b)$ modulo n.
24. Use the statement in the previous question to prove that a number is divisible by 11 if and only if the alternating sum of its digits (in base 10) is divisible by 11. Use this result to decide whether 13273425409265 is divisible by 11.

Good luck. Gábor Hetyei