Assignment 12

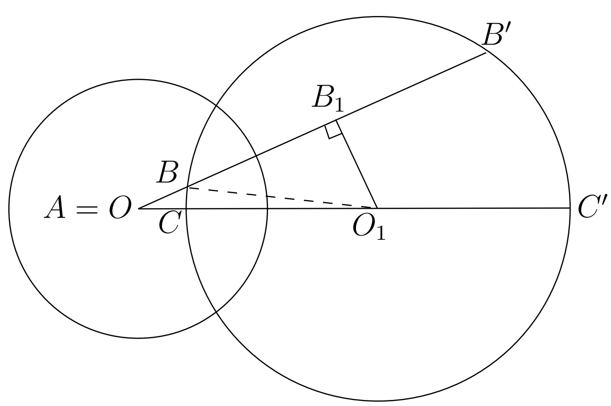
Oral questions

1. Assume $a, b, c \in \mathbb{R}$ satisfy $a^2 + bc = 1$, and let $T : \mathbb{C} \to \mathbb{C}$ be given by
$$T(z) = \frac{a\overline{z} + b}{c\overline{z} - a}.$$
Show that $T(T(z)) = z$ for all z. (All reflections of the Poincaré upper half plane model are represented by such a function.)

2. All hyperbolic rotations fixing the point i in the Poincaré upper half plane model are fractional linear transformations $z \mapsto \frac{az+b}{cz+d}$ sending i into i. Using this fact, and assuming that we have scaled our coefficients to satisfy $ad - bc = 1$, show that
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$
for some angle θ.

Question to be answered in writing

1. Complete the following proof of the *hyperbolic Pythagorean theorem* which states the following:
Any right triangle $\triangle ABC$ with $\angle C$ being the right angle satisfies $\cos(A) = \tanh(b)/\tanh(c)$.



Use the Poincaré disc model and assume that the vertex A is at the center of the disk. (The right angle of $ABC_\triangle$ is at C.) The lines AB and AC are represented by straight lines, the line BC is represented by an arc of a circle centered at O_1. Let B' resp. C' be the second intersection of OB resp OC with this circle and B_1 be the orthogonal projection of O to the line OB.

Using that the Euclidean distance OB equals $\tanh(c/2)$ and that $OB \cdot OB' = 1$ (justify why), prove that the Euclidean distance $BB' = 2/\sinh(c)$. Observe that the Euclidean distance CC' is similarly equal to $2/\sinh(b)$. Due to the Star Trek Lemma, the angle $\angle BO_1B_1$ is equal to $\angle B$. (Why?) Hence
$$\sin(B) = \frac{BB_1}{O_1B} = \frac{BB'}{2O_1C} = \frac{BB'}{CC'} = \frac{\sinh(b)}{\sinh(c)}.$$

Finally, using that $\cos(A) = AB_1/AO_1$, where $AB_1 = OB + BB'/2$ and $AO_1 = AC + CC'/2$, prove that

$$\cos(A) = \frac{\tanh(b)}{\tanh(c)}.$$