Sample Final Exam Questions.

The final exam will be comprehensive. The sample questions below address only material taught after Test 2. To prepare for questions about the earlier material, please refer to the questions on the sample tests. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. The questions below are sample questions related to stating and proving theorems. Besides trying to answer these questions, make sure you also review all homework exercises. The final exam may also have questions similar to those exercises.

In all sample questions F denotes a field.

1. Let F be a field and $p(x)$ be a nonzero polynomial in $F[x]$. Define congruence modulo $p(x)$ and prove it is an equivalence relation.
2. Given $p(x) \in F[x]$ as in the previous question, prove that congruence modulo $p(x)$ is compatible with the ring operations.
3. Assume $p(x) \in F[x]$ has degree n, where n is a positive integer. Prove that every congruence class modulo $p(x)$ may be represented by a polynomial of degree less than n, and show that this representative is unique. (We consider the constant 0 polynomial as a polynomial of degree $-\infty$.)
4. Assume $p(x) \in F[x]$ has positive degree. Prove that the set $F[x]/(p(x))$ of congruence classes is a commutative ring with identity that contains a subring that is isomorphic to F. (You may use the previous statement in your proof.)
5. Assume $p(x) \in F[x]$ has positive degree has positive degree and that $f(x) \in F[x]$ is relative prime to $p(x)$. Prove that the class of $f(x)$ is a unit in $F[x]/(p(x))$.
6. Assume $p(x) \in F[x]$ is an irreducible polynomial. Explain how the previous statement implies that $F[x]/(p(x))$ is a field.
7. Assume $p(x) \in F[x]$ is a polynomial of positive degree and that $F[x]/(p(x))$ is an integral domain. Prove that $p(x)$ is irreducible.
8. Assume $p(x) \in F[x]$ is an irreducible polynomial. Prove that the extension field $F[x]/(p(x))$ contains a root of $p(x)$.
9. Define an ideal, and prove that even integers are an ideal of $\mathbb{Z}$.
10. Assume R is a commutative ring with a multiplicative identity. Describe the ideal generated by a finite subset $\{c_1, \ldots, c_n\}$ of R. What is the name of an ideal generated of a single element?
11. Define congruence modulo an ideal and prove that it is an equivalence relation.

12. Prove that congruence modulo an ideal is compatible with the ring operations.

13. Let $f : R \to S$ be a ring homomorphism. Define the kernel of f, and prove that it is an ideal.

14. Let $f : R \to S$ be a ring homomorphism and $a, b \in R$. Describe, in terms of the kernel of f, when does $f(a)$ equal $f(b)$.

15. Prove that a ring homomorphism is injective if and only if its kernel is the zero ideal.

16. Let R be a ring and I an ideal of this ring. Describe the natural homomorphism from R to the ring R/I. (No need to prove that this is a homomorphism.) Prove that the natural homomorphism is surjective.

17. Let $a \in F$ be a fixed field element. Describe the kernel of the evaluation homomorphism $\phi_a : F[x] \to F$, sending each $f(x) \in F[x]$ into $f(a)$.

18. State the first isomorphism theorem.

19. State the third isomorphism theorem.

20. Define a prime ideal in a commutative ring with an identity and describe the prime ideals in $\mathbb{Z}$ and in $\mathbb{Q}[x]$.

Good luck. Gábor Hetyei