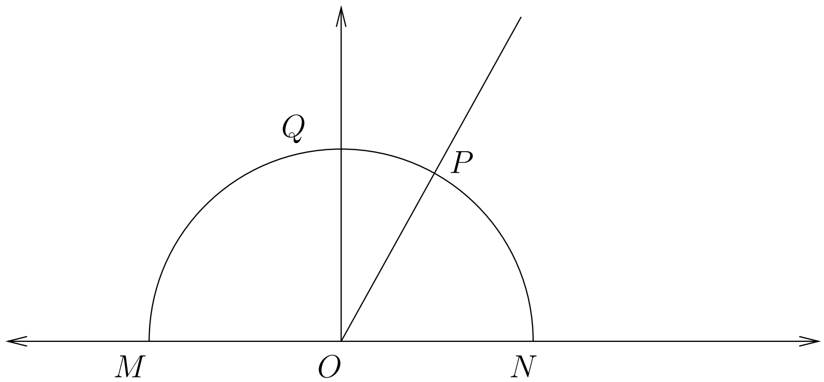
Hypercycles and horocycles in the Poincaré upper half plane model

We consider the Poincaré upper half plane model. We define a *hypercycle* as a set of all points located on the same side of a line, at the same perpendicular distance from the line.

Proposition 1 *For a vertical half-line in the Poincaré upper half plane model, the set of all hypercycles associated to this line is the set of all rays emanating from the intersection of the half-line with the real axis.*

Proof:



Without loss of generality we may assume that our vertical half-line emanates from the origin. The lines perpendicular to the line represented by this half line correspond to semicircles, centered at the origin O. Consider any point P in the upper half plane plane. To find its perpendicular projection to our line, we must draw a semicircle centered at O through P; the intersection Q of this semicircle with our vertical half-line is the perpendicular projection of P. Introducing M and N as the ideal points of the line represented by the semicircle, the hyperbolic distance of P and Q is

$$|\ln(PQMN)| = \left|\ln\left(\frac{PM}{MQ}\cdot\frac{NQ}{PN}\right)\right| = \left|\ln\left(\frac{PM}{PN}\right)\right|.$$

This quantity is constant exactly for all points P on a ray emanating from O ◇

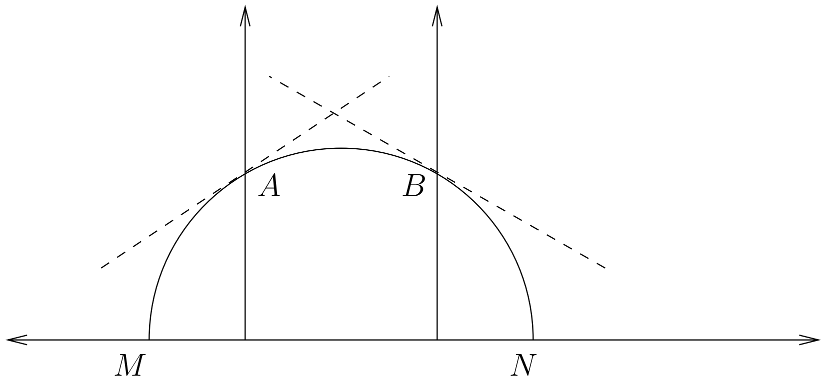
Proposition 2 *For a semicircle in the Poincaré upper half plane model, with ideal points M and N, the set of all hypercycles associated to this line is represented by the the set of all circular arcs contained in the upper half plane, whose endpoints are M and N.*

Proof: We may reduce this statement to Proposition 1 by performing an inversion, about the circle centered at M, of radius $|MN|$. (Here $|MN|$ is of course the Euclidean length.) This inversion takes the semicircle into a vertical line emanating from N. (The point M goes to infinity.) Since inversion is an isometry in the Poincaré upper half plane model, the representations of the hypercycles associated to the line represented by our semicircle go into the representations of the hypercycles associated to the line represented by the vertical line emanating from N. By Proposition 1, this second set of hypercycles is represented by all rays emanating from N. Under our inversion, these rays correspond to all circular arcs, containing both M and N. ◇

A horocycle is associated to an equivalence class of horoparallel lines. Two points A and B are on the same horocycle, if the horoparallel lines lines from the class through A and B have congruent angles with the line AB.

Proposition 3 *Consider the equivalence class of horoparallel lines, represented by all vertical half-lines. Then the horocycles associated to this family are represented by all horizontal lines in the upper half plane.*

Proof:



Consider a point A and a point B in the upper half plane. Recall that the line connecting A and B is represented by a semicircle, whose center is on the line of ideal points. By definition, the angle between this semicircle and the vertical half-line through A is the angle between the tangent to the semicircle A and the vertical line. The other angle at B is defined the same way. The two angles are equal exactly when the tangent lines at A and B are symmetric to the vertical axis of symmetry of the semicircle, which holds exactly when A and B are on the same horizontal line. $\diamond$

Proposition 4 *Consider the equivalence class of horoparallel lines, represented by all semicircles and one vertical line containing the ideal point M. Then the horocycles associated to this family are all circles contained in the upper half plane that touch the line of ideal points at M.*

Proof: In analogy of the proof of Proposition 2, this statement may be reduced to Proposition 3, by performing an inversion about any circle centered at M. This inversion takes our family, representing a class of horoparallel lines, into the family of all vertical half-lines. Since inversion preserves angles, the representations of horocycles go into the representations of the horocycles associated to the set of vertical half-lines. By Proposition 3, this corresponding family of horocycles is represented by all horizontal lines contained in the upper half plane. Under our inversion, these horizontal lines correspond to all circles that contain M, and are entirely contained in the upper half-plane, containing no other ideal point. These are then exactly those circles contained in the upper half plane that touch the ideal line at M. $\diamond$