Fractional linear transformations that are isometries of the Poincaré disk

A fractional linear transformation $z \mapsto \dfrac{az+b}{cz+d}$ that maps the unit disk $|z| < 1$ onto itself, must satisfy

$$\frac{az+b}{cz+d} \cdot \overline{\frac{az+b}{cz+d}} = 1 \quad \text{for all } z \in \mathbb{C} \text{ satisfying } |z| = 1.$$

Multiplying both sides by $(cz+d)(\overline{cz+d})$ yields

$$(az+b)(\overline{az+b}) = (cz+d)(\overline{cz+d}) \quad \text{for all } z \in \mathbb{C} \text{ satisfying } |z| = 1.$$

Using $z\overline{z} = 1$, after rearranging we obtain

$$|a|^2 + |b|^2 - |c|^2 - |d|^2 = (c\overline{d} - a\overline{b})z + (\overline{c}d - \overline{a}b)\overline{z} \quad \text{for all } z \in \mathbb{C} \text{ satisfying } |z| = 1.$$

Introducing $e := c\overline{d} - a\overline{b}$, the last equation may be rewritten as

$$|a|^2 + |b|^2 - |c|^2 - |d|^2 = 2|e| \cos\left(\arg(e) + \arg(z)\right) \quad \text{for all } z \in \mathbb{C} \text{ satisfying } |z| = 1.$$

The left hand side is constant, whereas the right hand side varies as $\arg(z)$ changes, unless $e = 0$. This yields the equations

$$|a|^2 + |b|^2 - |c|^2 - |d|^2 = 0 \quad \text{and} \tag{1}$$

$$\overline{c}d = \overline{a}b. \tag{2}$$

From here we distinguish to cases depending on whether $c = 0$.

Case 1: $c = 0$. Since $ad - bc \neq 0$, in this case we have $ad \neq 0$. Equation (2) implies $\overline{a}b = 0$, and since $ad \neq 0$, we must have $b = 0$. Equation (2) may be restated as $|a|^2 = |d|^2$. The map $z \mapsto \dfrac{az+b}{cz+d}$ is of the form $z \mapsto \dfrac{az}{d}$ where $\left|\dfrac{a}{d}\right| = 1$. We In this case our fractional linear transformation is a rotation about the origin.

Case 2: $c \neq 0$. Since we are allowed to multiply all of a, b, c, d by the same nonzero complex number, we may assume $c = 1$. Equation (2) may be restated as

$$d = \overline{a}b, \tag{3}$$

and substituting this and $c = 1$ into equation (1) we obtain $|a|^2 + |b|^2 - 1 - |a|^2|b|^2$, or, equivalently

$$(|a|^2 - 1)(|b|^2 - 1) = 0.$$

Since $ad - bc = a\overline{a}b - b = (|a|^2 - 1)b$ is not zero, the last highlighted equation implies

$$|b| = 1. \tag{4}$$

We may rewrite the map $z \mapsto \dfrac{az+b}{cz+d}$ a $z \mapsto \dfrac{a\frac{z}{b} + 1}{\frac{z}{b} + \overline{a}}$. This map may be described as the rotation about the origin $z \mapsto \dfrac{z}{b}$, followed by the map $z \mapsto \dfrac{az+1}{z+\overline{a}}$. This second map is supposed to take $z = 0$ into a point in the Poincaré disk, we must have $|1/\overline{a}| < 1$ or, equivalently, $|a| > 1$.

Lemma 1 *Given a nonzero complex number a of length greater than 1, the fractional linear transformation $z \mapsto \dfrac{az+1}{z+\overline{a}}$ may be described as the inversion about the circle centered at $-\overline{a}$, of radius $\sqrt{|a|^2-1}$, followed by a reflection about the vertical axis.*

Proof: The inversion is given by

$$z \mapsto -\overline{a} + \frac{a\overline{a}-1}{\overline{z}+a} = \frac{-\overline{a}(\overline{z}+a)+a\overline{a}-1}{\overline{z}+a} = \frac{-\overline{az}-1}{\overline{z}+a}.$$

Taking the negative of the conjugate of the end result amounts to a reflection about the vertical axis. Indeed, conjugation is a reflection about the horizontal axis, taking the negative amounts to a reflection about the origin, combining the two amounts to a reflection about the vertical axis. $\Diamond$

It should be noted that the circle centered at $-\overline{a}$, of radius $\sqrt{|a|^2-1}$, is orthogonal to the boundary of the Poincaré disk.

We may summarize our findings as follows.

Theorem 1 *Every fractional linear transformation that takes the Poincaré disk onto itself, may be written as a composition of at most two of the following maps:*

1. *a rotation about the center about the Poincaré disk;*
2. *an inversion about a circle that is perpendicular to the boundary of the Poincaré disk, followed by a reflection about the vertical axis.*

References

[1] D. Royster, "Non-Euclidean Geometry and a Little on How We Got There," Lecture notes, December 11, 2011.