Sample Test 2

This study guide is subject to updates until Thursday March 25.

The real test will have less questions and you will have about 75 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. Below I am only listing questions related to the definitions, theorems, and proofs I expect you to know. There will be also application exercises, similar to the already discussed homework questions.

1. Write a recurrence formula for the Stirling numbers $S(n,k)$ of the second kind using only two smaller Stirling numbers. Prove the formula and use it to provide a table for $S(n,k)$ for $n \leq 6$.

2. Define the Bell numbers and express them in terms of the Stirling numbers of the second kind.

3. What is the number of onto functions from $[n]$ to $[m]$? Express your answer in terms of the Stirling numbers of the second kind. Justify your answer.

4. Explain why the Stirling numbers of the second kind satisfy the identity

$$S(n,k) = \sum_{j=0}^{n} \binom{n-1}{j} S(n-j-1, k-1) \quad \text{for } n \geq 1 \text{ and } k \geq 1.$$

5. Give a combinatorial proof of the formula

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k$$

for positive integer substitutions into x.

6. Write a recurrence formula for the integer partition numbers $P(n,k)$ and provide a table for them for $n \leq 6$. (Hint: the formula is $P(n,k) = \sum_{j=1}^{k} P(n-k, j)$. If I ask such a question, I will not give this hint on the test.)

7. Prove the equation $P(n,k) = \sum_{j=1}^{k} P(n-k, j)$.

8. Prove that the number of partitions of n into k parts is the same as the number of partitions of n with largest part of size k. (Hint: use Ferrers diagrams and transpose them. I will not give this hint on the test.)

9. What kind of functions are counted by the partition number $P(n,k)$? (What is the size of the domain, the target, are the elements distinct or identical, are the functions one-to-one or onto?)

10. State and prove the inclusion-exclusion formula.

11. n persons attend a party. A fire breaks out in the building, while outside there is a heavy rain. Everybody rushes to the wardrobe, picks up an umbrella, and leaves. What is the probability that no one picked their own umbrella? Give an exact formula as a function of n and an approximate number for large n. Justify your answer.

12. Prove by induction that
$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
holds for every nonnegative integer n.

13. The Lucas numbers $L_0, L_1, \ldots, L_n, \ldots$ are given by the initial conditions $L_0 = 2$ and $L_1 = 1$, and by the recurrence $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Prove by induction that $L_n < 2^n$ holds for $n \geq 1$.

14. Find $\binom{1/3}{5}$.

15. What is the coefficient of x^m in $\dfrac{1}{(1-x)^n}$? Justify your answer!

16. Write $1 + x + x^2 + \cdots + x^n$ and $\sum_{n=0}^{\infty} x^n$ in closed form.

17. Find the coefficient of x^k in $\dfrac{2x}{3+5x}$.

18. State the convolution formula expressing the coefficient of x^k in $f(x) \cdot g(x)$ where $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$.

19. Write the convolution formula for the exponential generating functions of two sequences of numbers. Explain how this follows from the ordinary convolution formula.

20. There are 3 questions on a quiz, the first worth 3 points, the second 2 points, the third 4 points. Write the ordinary generating function for number a_n of ways to make n points on the quiz.

21. Give a closed form formula for the sequence a_n given by $a_0 = 7$, $a_1 = 16$, and $a_{n+1} = 5a_n - 6a_{n-1}$. (Solution is $a_n = 5 \cdot 2^n + 2 \cdot 3^n$, but you have to show it.)

Good luck. Gábor Hetyei