Properties of an inversion

The following presentation of the properties of the inversion operation may be found in the Hungarian language textbook written by György Hajós [1], cited in the references.

Proposition 1 *The inversion about the circle given by $x^2 + y^2 = 1$ takes the point $(x, y) \neq (0, 0)$ into the point $(x/(x^2 + y^2), y/(x^2 + y^2))$.*

The proof is left as an exercise.

Corollary 1 *The above inversion takes the curve given by the equation $F(x, y) = 0$ into the curve given by the equation $F(x/(x^2 + y^2), y/(x^2 + y^2)) = 0$.*

Theorem 1 *Inversion takes each line or a circle into a line or circle.*

Proof: It suffices to prove the statement for the inversion about the circle given by $x^2 + y^2 = 1$. The equation of a line or circle may be written in the form

$$A(x^2 + y^2) + Bx + Cy + D = 0, \tag{1}$$

where not all of A, B, C and D are zero. By Corollary 1, the result of the inversion is the curve given by the equation

$$A\left(\left(\frac{x}{x^2 + y^2}\right)^2 + \left(\frac{y}{x^2 + y^2}\right)^2\right) + B\frac{x}{x^2 + y^2} + C\frac{y}{x^2 + y^2} + D = 0.$$

Multiplying both sides by $x^2 + y^2$ yields the equation

$$A + Bx + Cy + D(x^2 + y^2) = 0, \tag{2}$$

which again is the equation of a line or a circle. $\diamond$

Remark 1 The above proof omits the observation that an equation of the form (1) may describe the empty set, e.g., $x^2 + y^2 + 1 = 0$ is satisfied by no point in the plane. Fortunately it is clear that inversion takes a nonempty set of points into a nonempty set of points.

Theorem 2 *Theorem 1 may be refined as follows:*

   (i) The inverse of a line containing the pole of inversion is a line of the same kind.
   (ii) The inverse of a circle not containing the pole of the inversion is a circle of the same kind.
   (iii) The inverse of a circle containing the pole of inversion is a line not containing the pole of the inversion (and vice versa).

Proof: We only need to have a closer look at he equation (1). It describes a line containing $(0,0)$ exactly when $A = D = 0$, i.e. when the constant term, as well as the coefficient of x^2 is zero. When this holds, the inverse curve, given by equation (2), has the same property. This concludes the proof of item (i).

Equation (1) describes a circle not containing the origin, exactly when the coefficient of x^2 and the constant term are both nonzero. When this holds, the inverse curve, given by equation (2), has again the same property. Thus item (ii) is true as well.

Observe finally that the curve given by equation (1) is a circle containing the origin exactly when the coefficient of x^2 is nonzero but the constant term is zero, whereas equation (1) describes a line not containing the origin exactly when the coefficient of x^2 is zero but the constant term is nonzero. Inversion about a circle whose pole is $(0,0)$ takes a curve of the first kind into a curve of the second kind and vice versa. ◊

Proposition 2 *If a circle touches another circle or a line in a point that is different from the pole of inversion, then the inverse curves also touch each other in a point that is different from the pole of inversion.*

Proof: The inverse curves can not intersect at the pole of the inversion, because at least one of the original curves is a circle. If the inverse curves have more than one point in common then the same holds for the original curves which is impossible: a circle, touched by a circle or a line, has only one point in common with the other curve. Therefore the inverse curves have exactly one point in common: the inverse of the common point of the original curves. At least one of the inverse curves is not a line since at least one of the original curves does not contain the pole of the inversion. When a circle and a circle or line have exactly one point in common, they touch each other. ◊

Corollary 2 *Given a line ℓ and a point $P \in \ell$ let $\mathcal{F}(\ell, P)$ be the set of curves consisting of ℓ and of all circles touching ℓ at P. An inversion whose pole is not P takes the family of curves $\mathcal{F}(\ell, P)$ into a similar family $\mathcal{F}(\ell', P')$. Here P' is the inverse of P.*

Proposition 3 *Let ℓ_1 and ℓ_2 be two lines that intersect in a point P. An inversion, whose pole is not P takes these lines into two circles whose angle at their intersection is the same as the angle of ℓ_1 and ℓ_2.*

The proof is left as an exercise.

Theorem 3 *If two sets, each of which is a line or a circle intersect in a point that is different from the pole of the inversion, then their inverses intersect each other in a point that is different from the pole of the inversion, at the same angle.*

Proof: Let P be a common point of the two sets that is different from the pole of the inversion. Let ℓ_1 be the first set if that set is a line, otherwise let ℓ_1 be the tangent of the first set at P. Define the line ℓ_2 similarly. Consider the families of circles and lines $\mathcal{F}(\ell_1, P)$ and $\mathcal{F}(\ell_2, P)$. These families have the following properties: we may replace the first set with any line or circle from $\mathcal{F}(\ell_1, P)$ and the second set with any line or circle from $\mathcal{F}(\ell_2, P)$ and the resulting two sets will intersect in the same point P, at the same angle, which is also the angle of ℓ_1 and ℓ_2. By Corollary 2, the families $\mathcal{F}(\ell_1, P)$ and $\mathcal{F}(\ell_2, P)$ are taken into similar families $\mathcal{F}(\ell_1', P')$ and $\mathcal{F}(\ell_2', P')$, where P' is the inverse of P. We may take our first set to be any element of $\mathcal{F}(\ell_1', P')$ and our second set to be any element of $\mathcal{F}(\ell_2', P')$, the two selected sets have the same angle at P'. Thus the inverses of our original sets may be replaced with the inverses of ℓ_1 and ℓ_2. The statement now follows from Proposition 3. $\diamond$

References

[1] György Hajós, *"Bevezetés a geometriába,"* Tankönyvkiadó, Budapest, 1960.