Newton sums

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1 Formal power series

For any function $f : \mathbb{N} \longrightarrow \mathbb{R}$ consider the generating function

$$F(t) = \sum_{n \geq 0} f(n) \cdot t^n.$$

We consider these generating functions as *formal power series*, i.e., we add, subtract, multiply an divide them formally, without expecting convergence. Whenever we write

$$\frac{1}{1-t} = \sum_{n \geq 0} t^n$$

we mean that the formal power series 1 divided by the formal power series $1-t$ yields the formal power series on the right hand side. There is no limitation on the use of addition, subtraction, multiplication of formal power series, there is some on division. For instance $\frac{1}{t}$ is not a formal power series any more. We may divide the formal power series $F(t)$ by the formal power series $G(t)$ if the lowest degree of t occurring in $F(t)$ is higher than the lowest degree of t occurring in $G(t)$.

Finally let us note that we may take also the formal derivative of a formal power series. The rule is

$$\frac{d}{dt} \sum_{n \geq 0} f(n) \cdot t^n = \sum_{n \geq 0} n \cdot f(n) \cdot t^{n-1}$$

It can be shown that the usual rules on the derivatives of sums, differences, product, quotients, and even the chain rule apply to formal derivatives of formal power series. (For this last operation we need to define $F(G(t))$ which is only possible when $G(t) = \sum_{n \geq 0} g(n) \cdot t^n$ satisfies $g(0) = 0$.)

2 Finite differences

Definition 1 *Given a function* $f : \mathbb{N} \longrightarrow \mathbb{R}$, *we define the* backward difference Δf *by*

$$\Delta f(n) = \begin{cases} f(n) - f(n-1) & \text{if } n \geq 1, \\ f(0) & \text{if } n = 0. \end{cases}$$

In other words, we assume $f(-1) = 0$.

The generating function $F(t)$ is of $f : \mathbb{N} \longrightarrow \mathbb{R}$ and the generating function $G(t)$ of Δf are connected by the formula

$$G(t) = (1-t) \cdot F(t) \tag{1}$$

In fact,

$$G(t) = \sum_{n \geq 0} (f(n) - f(n-1)) \cdot t^n = \sum_{n \geq 0} f(n) \cdot t^n - \sum_{n \geq 0} f(n) \cdot t^{n+1} = (1-t)F(t).$$

3 Newton sums

Definition 2 *The* Newton sums $S_0, S_1, \ldots : \mathbb{N} \longrightarrow \mathbb{R}$ *are defined by*

$$S_k(n) = 1^k + 2^k + \ldots + n^k.$$

In particular $S_0(n) = n$.

The finite difference of S_k is the function n^k. This justifies the formula

$$\Delta S_{k+1}(n) = n \cdot \Delta S_k(n) \tag{2}$$

since $n^{k+1} = n \cdot n^k$. Let us denote the generating function of S_k by F_k, i.e., let us set

$$F_k(t) = \sum_{n \geq 0} S_k(n) \cdot t^n.$$

Proposition 1 *The formal power series* $F_0(t), F_1(t), \ldots$ *satisfy the recursion formula*

$$F_{k+1}(t) = \frac{t}{1-t} \cdot \frac{d}{dt} \left((1-t) \cdot F_k(t) \right).$$

Proof: Using equations (1) and (2) we may write

$$\begin{aligned}
(1-t)F_{k+1}(t) &= \sum_{n\geq 0} \Delta S_{k+1}(n)\cdot t^n = \sum_{n\geq 0} n\cdot \Delta S_k(n)\cdot t^n = t\cdot \sum_{n\geq 1} n\cdot \Delta S_k(n)\cdot t^{n-1} \\
&= t\cdot \frac{d}{dt}\sum_{n\geq 1} \Delta S_k(n)\cdot t^n = t\cdot \frac{d}{dt}\left((1-t)\cdot F_k(t)\right)
\end{aligned}$$

Dividing both sides by $(1-t)$ yields the stated formula. $\diamond$

Proposition 1 allows us to recursively compute the formal power series $F_k(t)$. To begin,

$$F_0(t) = \sum_{n\geq 0} n\cdot t^n = t\cdot \sum_{n\geq 0} n\cdot t^{n-1} = t\cdot \frac{d}{dt}\sum_{n\geq 0} t^n = t\cdot \frac{d}{dt}\left(\frac{1}{1-t}\right) = \frac{t}{(1-t)^2}$$

Next we obtain

$$F_1(t) = \frac{t}{1-t}\cdot \frac{d}{dt}\left((1-t)\cdot F_0(t)\right) = \frac{t}{1-t}\cdot \frac{d}{dt}\left(\frac{t}{1-t}\right) = \frac{t}{1-t}\cdot \frac{1-t+t}{(1-t)^2} = \frac{t}{(1-t)^3}$$

and

$$F_2(t) = \frac{t}{1-t}\cdot \frac{d}{dt}\left((1-t)\cdot F_1(t)\right) = \frac{t}{1-t}\cdot \frac{d}{dt}\left(\frac{t}{(1-t)^2}\right) = \frac{t}{1-t}\cdot \frac{(1-t)^2+2\cdot t\cdot (1-t)}{(1-t)^4} = t\cdot \frac{1+t}{(1-t)^4}$$

We may observe that all formal power series are the quotients of some polynomial and of some power of $(1-t)$. Observe also that the polynomial in the numerator is divisible by t. Let us prove this.

Proposition 2 *For all k,*

$$F_k(t) = \frac{t\cdot p_k(t)}{(1-t)^{k+2}}$$

holds, where the polynomials $p_k(t)$ are given by $p_0(t)=1$ and the recursion formula

$$p_{k+1}(t) = (p_k(t) + t\cdot p_k'(t))\cdot (1-t) + (k+1)\cdot t\cdot p_k(t).$$

Proof: We proceed by induction on t. For $k=0$ the statement holds, as seen above. Using Proposition 1 and the induction hypothesis we may write

$$\begin{aligned}
F_{k+1}(t) &= \frac{t}{1-t}\cdot \frac{d}{dt}\left((1-t)\cdot F_k(t)\right) = \frac{t}{1-t}\cdot \frac{d}{dt}\left(\frac{t\cdot p_k(t)}{(1-t)^{k+1}}\right) \\
&= \frac{t}{1-t}\cdot \frac{(p_k(t)+t\cdot p_k'(t))\cdot (1-t)^{k+1} + t\cdot p_k(t)\cdot (k+1)\cdot (1-t)^k}{(1-t)^{2k+2}} \\
&= t\cdot \frac{(p_k(t)+t\cdot p_k'(t))\cdot (1-t) + (k+1)\cdot t\cdot p_k(t)}{(1-t)^{k+3}}
\end{aligned}$$

$\diamond$

We conclude this section with a table for the polynomials $p_k(t)$

k	$p_k(t)$
0	1
1	1
2	$1+t$
3	$1+4t+t^2$
4	$1+11t+11t^2+t^3$
5	$1+26t+66t^2+26t^3+t^4$

4 Binomial series

In order to obtain a formula for the Newton sums $S_k(n)$ from the formulas for their generating functions, we need to look at *binomial series* $(1+t)^\alpha$ for negative integer α's. Actually, such a series may be defined for every real alpha, by using the following extended definition of the binomial coefficients.

Definition 3 *Given $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$ we define the binomial coefficient $\binom{\alpha}{n}$ by*

$$\binom{\alpha}{n} = \frac{\alpha \cdot (\alpha-1) \cdots (\alpha-n+1)}{n!}$$

Using this definition we may write

$$(1+t)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} t^n$$

It may be shown that this definition is consistent with the operations on formal power series. For example for $\alpha = -1$ we get

$$\binom{-1}{n} = \frac{(-1)\cdot(-2)\cdots(-n)}{n!} = (-1)^n$$

and

$$\frac{1}{1+t} = \sum_{n\geq 0} (-1)^n \cdot t^n.$$

Considering the special form of the generating functions $F_k(t)$ we are interested in the formula for $(1-t)^{-k-2}$.

Lemma 1 *For any positive integer k,*

$$(1-t)^{-k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} \cdot t^n$$

holds.

Proof: By definition we have

$$\binom{-k}{n} = \frac{(-k)\cdot(-k-1)\cdots(-k-n+1)}{n!} = (-1)^n \cdot \frac{k\cdot(k+1)\cdots(n+k-1)}{n!} = (-1)^n \binom{n+k-1}{n}$$

and by the formula $\binom{m}{n} = \binom{m}{m-n}$ we obtain

$$\binom{-k}{n} = (-1)^n \binom{n+k-1}{k-1}$$

Hence we have

$$(1-t)^{-k} = \sum_{n\geq 0} \binom{-k}{n} (-1)^n \cdot t^n = \sum_{n\geq 0} \binom{n+k-1}{k-1} \cdot t^n$$

$\diamond$

To summarize, using Proposition 1 and Lemma 1 we may write

$$F_k(t) = t \cdot p_k(t) \cdot \sum_{n\geq 0} \binom{n+k+1}{k+1} \cdot t^n = p_k(t) \cdot \sum_{n\geq 0} \binom{n+k+1}{k+1} \cdot t^{n+1}$$

and so we have

$$F_k(t) = p_k(t) \cdot \sum_{n\geq 1} \binom{n+k}{k+1} \cdot t^n. \tag{3}$$

For $k=0$ we get

$$F_0(t) = 1 \cdot \sum_{n\geq 0} \binom{n}{1} \cdot t^n = \sum_{n\geq 0} \binom{n}{1} \cdot t^n,$$

and

$$S_0(n) = n.$$

For $k=1$ we get

$$F_1(t) = 1 \cdot \sum_{n\geq 1} \binom{n+1}{2} \cdot t^n,$$

and

$$S_1(n) = \binom{n+1}{2} = \frac{n\cdot(n-1)}{2}.$$

For $k=2$ we get

$$F_2(t) = (1+t) \cdot \sum_{n\geq 1} \binom{n+2}{3} \cdot t^n,$$

and

$$S_2(n) = \binom{n+2}{3} + \binom{n+1}{3} = \frac{(n+2)(n+1)n + (n+1)n(n-1)}{6} = \frac{n(n+1)(2n+1)}{6}.$$

For $k = 3$ we get

$$F_3(t) = (1 + 4t + t^2) \cdot \sum_{n \geq 1} \binom{n+3}{4} \cdot t^n,$$

and

$$\begin{aligned} S_3(n) &= \binom{n+3}{4} + 4 \cdot \binom{n+2}{4} + \binom{n+1}{4} \\ &= \frac{(n+3)(n+2)(n+1)n + 4(n+2)(n+1)n(n-1) + (n+1)n(n-1)(n-2)}{24} \\ &= \frac{n(n+1)(6n^2+6n)}{24} = \frac{n^2(n+1)^2}{4}. \end{aligned}$$

For $k = 4$ we get

$$F_4(t) = (1 + 11t + 11t^2 + t^3) \cdot \sum_{n \geq 1} \binom{n+4}{5} \cdot t^n,$$

and

$$\begin{aligned} S_4(n) &= \binom{n+4}{5} + 11 \cdot \binom{n+3}{5} + 11 \cdot \binom{n+2}{5} + \binom{n+1}{5} \\ &= \frac{(n+4)(n+3)(n+2)(n+1)n + 11(n+3)(n+2)(n+1)n(n-1)}{120} \\ &\quad + \frac{11(n+2)(n+1)n(n-1)(n-2) + (n+1)n(n-1)(n-2)(n-3)}{120} \\ &= \frac{24n^5 + 60n^4 + 40n^3 - 4n}{120} = \frac{4n(2n+1)(n+1)(3n^2+3n-1)}{120}. \end{aligned}$$

Finally, for $k = 5$ we get

$$F_5(t) = (1 + 26t + 66t^2 + 26t^3 + t^4) \cdot \sum_{n \geq 1} \binom{n+5}{6} \cdot t^n,$$

and

$$\begin{aligned} S_5(n) &= \binom{n+5}{6} + 26 \cdot \binom{n+4}{6} + 66 \cdot \binom{n+3}{6} + 26 \cdot \binom{n+2}{6} + \binom{n+1}{6} \\ &= \frac{(n+5)(n+4)(n+3)(n+2)(n+1)n + 26(n+4)(n+3)(n+2)(n+1)n(n-1)}{720} \\ &\quad + \frac{66(n+3)(n+2)(n+1)n(n-1)(n-2) + 26(n+2)(n+1)n(n-1)(n-2)(n-3)}{720} \\ &\quad + \frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{720} \\ &= \frac{-60n^2 + 300n^4 + 360n^5 + 120n^6}{720} = \frac{n^2(2n^2+2n-1)(n+1)^2}{12}. \end{aligned}$$