Axioms for real numbers

a. $\forall x \forall y \forall z\ (x+y)+z = x+(y+z)$ (Associative Law for Addition).
b. $\forall x \forall y\ x+y = y+x$ (Commutative Law for Addition).
c. There is a zero element 0 such that $x+0 = x$ holds for all x (Identity Law for Addition).
d. For every x there is a $-x$ such that $x+(-x) = 0$ holds (Inverses Law for Addition).

So far: with respect of addition we have an *Abelian group*.
For example, $\mathbb{Z}_2$ with respect to addition is an Abelian group.

e. $\forall x \forall y \forall z\ (xy)z = x(yz)$ (Associative Law for Multiplication). With respect to multiplication we have a *semigroup*.
i. $\forall x \forall y \forall z\ x(y+z) = xy+xz$ (Distributive Law). If we do not postulate the commutativity of the multiplication, then this is the Left Distributive Law, and we also need to postulate the right distributive law, which is $\forall x \forall y \forall z\ (y+z)x = yx+zx$.

So far: we have a *ring*.
For example the set of 2×2 matrices with integer entries is a ring.

f. $\forall x \forall y\ xy = yx$ (Commutative Law for Multiplication).
g. There is a multiplicative identity 1 such that $x \cdot 1 = x$ holds for all x (Identity Law for Multiplication).
h. Each $x \neq 0$ has a multiplicative inverse x^{-1} satisfying then $xx^{-1} = 1$ (Inverses Law for Multiplication).
n. $0 \neq 1$ (Non-triviality).

So far: we have a *field*.
For example, $\mathbb{Z}_p$ is a field for any prime p.

j. For all x and y, precisely one of $x < y$ or $x = y$ or $x > y$ holds (Trichotomy Law). (This includes that the relation $<$ is *irreflexive* and *antisymmetric*.)
k. For all x, y and z, iff $x < y$ and $y < z$, then $x < z$ (Transitive Law).

Axioms j and k postulate that $<$ is a *total order*. If we do not require trichotomy, only the irreflexive property $\forall x\ x \not< x$ and antisymmetry ($x < y$ and $y < x$ can not hold simultaneously), then we have a *partial order*. For example, $\mathbb{Z}$ is a totally ordered set.

l. For all x, y and z, if $x < y$ then $x + z$ ¡ y+z (Addition Law for Order).
m. For all x, y and z, if $x < y$ and $z > 0$, then $xz < yz$ (Multiplication Law for Order).

The last two axioms guarantee that the relation $<$ is *compatible* with the operations.
So far we have an *ordered field*.
For example $\mathbb{Q}$ is an ordered field but the set $\{x \in \mathbb{Q} : x^2 \leq 2\}$ is bounded from above but has no least upper bound.

The ordered field F satisfies the *Least Upper Bound Property* if every non-empty subset of F that is bounded above has a least upper bound. (Real numbers have this property.)