Study guide for the final exam

This study guide is subject to updates until the last day of classes.

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The final exam will be cumulative. Regarding the material covered before our second test, please refer to the study guides to our tests. This study guide helps only with the material covered after our second test. The final exam will last 150 minutes. You will have to give the simplest possible answer and show all your work. The questions below are sample questions. Besides trying to answer these questions, make sure you also review all homework exercises. The final exam may also have questions similar to those exercises. During the exam, the usage of books or notes, or communicating with other students will not be allowed.

1. Suppose $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of the same matrix A, and let λ and μ be the corresponding eigenvalues. Complete the following sentences with $\lambda = \mu$ or $\lambda \neq \mu$.
   (a) If ... then $\mathbf{u}$ and $\mathbf{v}$ must be linearly independent.
   (b) If ... then any linear combination of $\mathbf{u}$ and $\mathbf{v}$ is also an eigenvector.
2. Which of the following sets is a vector space? Explain why.
   (a) polynomials of degree 4;
   (b) polynomials of degree at most 4;
   (c) polynomials with constant term 0;
   (d) polynomials with constant term 2;
   (e) vectors in the first quadrant of $\mathbb{R}^2$.
3. Consider the linear transformation that takes each polynomial of degree at most 4 into its constant term. Describe the kernel (nullspace) and range of this linear transformation. What are the dimensions of these vector spaces?
4. Find a basis for the range and for the nullspace of the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ whose matrix in the standard basis is $[T] = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.
5. Find a basis for the vector space spanned by the columns of $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ by selecting a maximal linearly independent subset.
6. The first two rows of a 3×4 matrix are linearly independent, but the third row is a linear combination of the first two rows. What is the rank of the matrix?

7. Write the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of the vector $[\mathbf{x}] = \begin{bmatrix}1\\2\end{bmatrix}$ in the basis $\mathcal{B} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$.

8. Find the standard coordinate vector $[\mathbf{x}]$ if the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ in the basis $\mathcal{B} = \left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\0\\0\end{bmatrix}\right\}$ is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}-1\\0\\6\end{bmatrix}$.

9. Explain why the set $\mathcal{B} = \{1, x-2, x^2+1\}$ is a basis of the vector space $\mathbb{P}_2$ of polynomials of degree at most 2 and write down the $\mathcal{B}$-coordinates of $x^2 + x$.

10. Find a basis for the vector space $\left\{\begin{bmatrix}s+t\\s-t+2r\\2s+3t-r\end{bmatrix} : s, r, t \in \mathbb{R}\right\}$.

11. The matrix of a linear transformation in the standard basis is $A = \begin{bmatrix}1 & 1\\-1 & 2\end{bmatrix}$. Write the matrix of the same linear transformation in the basis $\mathcal{B} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$. (Use the change of basis formula.)

12. Write the matrix of the reflection about the line $y = -x$.

13. A square matrix B is similar to the matrix A. Is B invertible if 0 is an eigenvalue of A?

14. A square matrix is similar to diagonal matrix $D = \begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{bmatrix}$. What are the dimensions of the eigenspaces?

15. Diagonalize the matrix $A = \begin{bmatrix}-1 & 2\\-6 & 6\end{bmatrix}$, that is, write it in the form $A = P \cdot D \cdot P^{-1}$ where D is a diagonal matrix and P is and invertible matrix. Give a reason if this is not possible.

16. Diagonalize the matrix $A = \begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}$, that is, write it in the form $A = P \cdot D \cdot P^{-1}$ where D is a diagonal matrix and P is and invertible matrix. Give a reason if this is not possible.

17. Find A^n for the matrix $A = PDP^{-1}$ where $D = \begin{bmatrix}-1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{bmatrix}$, $P = \begin{bmatrix}1 & 2 & 0\\0 & 1 & 0\\0 & 0 & 3\end{bmatrix}$ and $P^{-1} = \begin{bmatrix}1 & -2 & 0\\0 & 1 & 0\\0 & 0 & 1/3\end{bmatrix}$.

18. The sequence $a_0, a_1, \ldots, a_n$ satisfies the recurrence $a_{n+2} = 5a_{n+1} - 6a_n$ for $n \geq 0$ and the initial conditions $a_0 = 1$, $a_1 = 4$. Write a closed form formula for a_n.

Good luck. Gábor Hetyei