From Efron's coins to alternation acyclic tournaments

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Friday, October 16, 2020
Combinatorics, Algebra and Geometry Seminar
George Mason University

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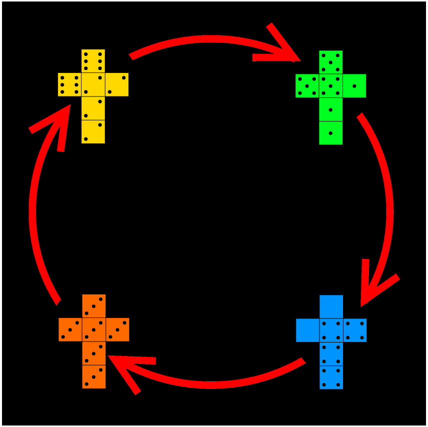

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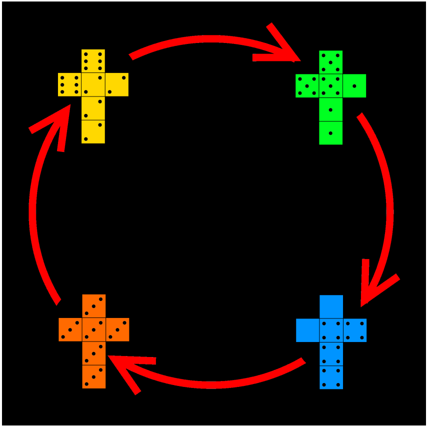

Dice defeat each other in cyclic order

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See Stearns (at least $0.55n/\log(n)$ voters), Erdős and Moser ($O(n/\log(n))$ voters), and Bednay-Bozóki (dice with $\lfloor 6n/5 \rfloor$ faces) for improved results.

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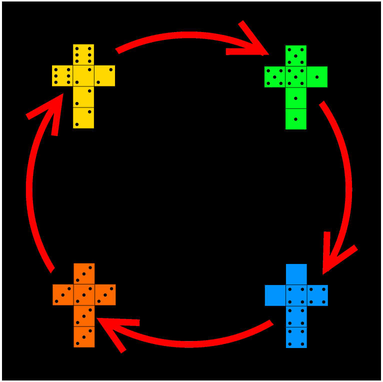

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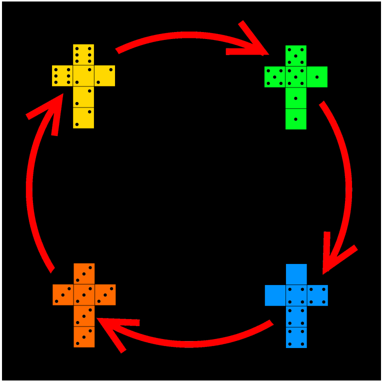
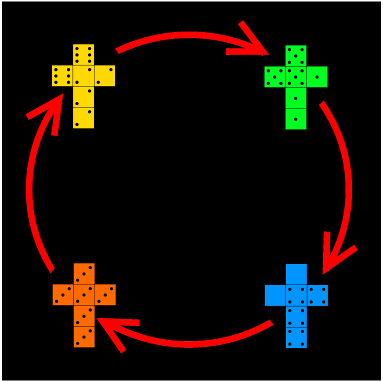


Each dice displays at most 2 values

Efron's dice could be coins



Question arises: which tournaments can be represented by (unfair) coins?

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- We may assume $a < b$ for all coin types. (This is a lemma!)

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Relation between (a_i, b_i) and (a_j, b_j)	$i \to j$ exactly when
$(a_i, b_i) = (a_j, b_j)$	$x_i > x_j$
$a_i = a_j < b_i < b_j$	$1/x_j > 1/x_i + 1$
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$a_i < a_j < b_i = b_j$	$x_i > x_j + 1$
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The fourth line should look familiar, if you saw the *Linial arrangement*.

The Linial arrangement

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$\mathcal{L}_{n-1}$ is given by

$$x_i - x_j = 1 \quad \text{where} \quad 1 \leq i < j \leq n$$

in the $(n-1)$-dimensional vector space
$V_{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$.

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To each region R in $\mathcal{L}_{n-1}$ we may associate a tournament on $\{1, \ldots, n\}$ as follows: for each $i < j$ we set $i \to j$ if $x_i > x_j + 1$ and we set $j \to i$ if $x_i < x_j + 1$.

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Proposition (Postnikov-Stanley, Shmulik Ravid)

A tournament T on $\{1, \ldots, n\}$ corresponds to a region R in $\mathcal{L}_{n-1}$ if and only if T is semiacyclic.

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- An *ascending cycle* is a cycle in which the number of descents does not exceed the number of ascents.
- A (labeled) tournament is *semiacyclic* if it does not contain an ascending cycle.

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Theorem

Assume a set of n winner and fair coins is listed in increasing lexicographic order of their types. If the domination graph is a tournament, it must be semiacyclic. Conversely every semiacyclic tournament is the domination graph of a set of winner coins.

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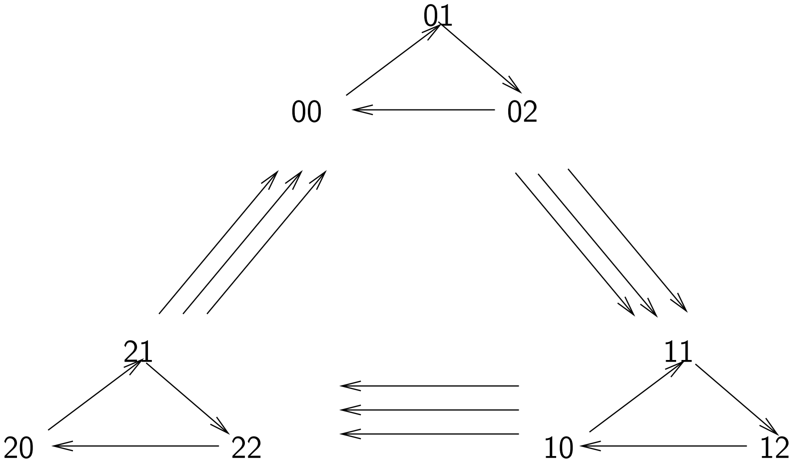

$C_3 \times C_3$ has no semiacyclic labeling

$C_3 \times C_3$ is representable using coins of both kinds

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General sets of coins

Analogous results hold for loser and fair coins. (Reverse arrows or replace each x_i with $1/x_i$.)

Corollary

If a tournament T may be represented as the dominance graph of a system of coins, then its vertex set V may be written as a union $V = V_1 \cup V_2$, such that the full subgraphs induced by V_1 and V_2, respectively, may be labeled to become semiacyclic tournaments.

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Theorem

Suppose the tournaments T_1 and T_2 have the property that they are not semiacyclic for any ordering of their vertex sets. Then the tournament $T_1 \times T_2$ can not be the dominance graph of any system of coins.

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Regarding semiacyclic tournaments

Postnikov also used partial differential equations and implicit function equations to show that the number of alternating trees is

$$2^{-n}\sum_{k=0}^{n}\binom{n}{k}(k+1)^{n-1}.$$

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Counting semiacyclic tournaments directly would be desirable.

Alternation acyclic tournaments

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A directed cycle $C = (c_0, c_1, \ldots, c_{2k-1})$ is *alternating* if ascents and descents alternate along the cycle, that is, $c_{2j} \stackrel{d}{\to} c_{2j+1}$ and $c_{2j+1} \stackrel{a}{\to} c_{2j+2}$ hold for all j (here we identify all indices modulo $2k$).

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Theorem

Suppose a tournament T on $\{1, \ldots, n\}$ contains a closed alternating walk $(c_0, c_1, \ldots, c_{2k-1})$, that is, a closed walk, in which descents and ascents alternate. Then T contains an alternating cycle of length 4.

In a tournament T on $\{1, \ldots, n\}$, there is a *right-alternating walk* from u to v if $u = v$ or there is a directed walk $u = w_0 \xrightarrow{d} w_1 \xrightarrow{a} w_2 \xrightarrow{d} \cdots \xrightarrow{d} w_{2i-1} \xrightarrow{a} w_{2i} = v$ from u to v in which descents and ascents alternate, the first edge being a descent and the last edge being an ascent. We will use the notation $u \leq_{ra} v$ when there is a right-alternating walk from u to v, and we will refer to $\leq_{ra}$ as the *right-alternating walk order induced by* T. We will also use the shorthand notation $u <_{ra} w$ when $u \leq_{ra} v$ and $u \neq v$ hold.

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Proposition

A tournament T on $\{1,\ldots,n\}$ is alternation acyclic, if and only the induced right-alternating walk order is a partial order.

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Definition

Given a permutation $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, we will say that the *labeling induced by the positions in* π is the labeling that associates to each $i \in \{1, 2, \ldots, n\}$ the position $\pi^{-1}(i)$ of i in π.

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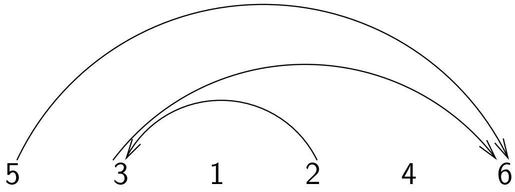

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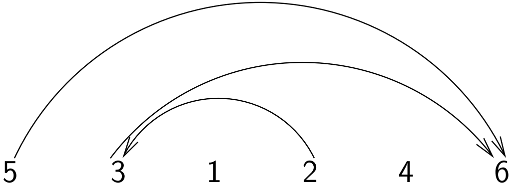

The arrows represent $i \to p(i)$, $\pi = 531246$.

The biordered forest representation

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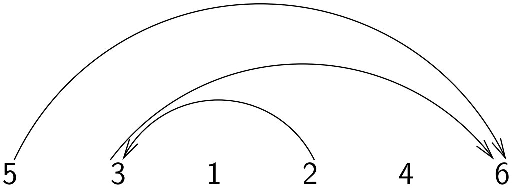

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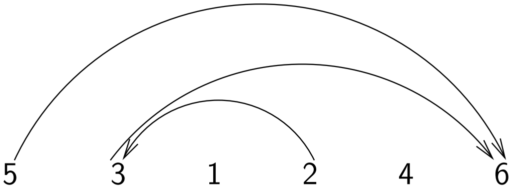

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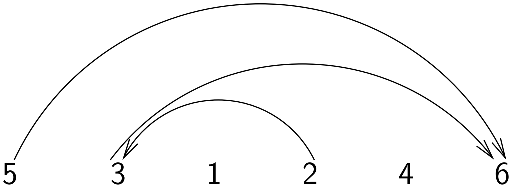

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$\pi(2) = 3$. The number 3 the leftmost number larger than 2 for which $2 \xrightarrow{a} 3$. All numbers larger than 2 that are to the left of 3 defeat 2, and 2 defeats all numbers larger than 2 to the right of 3. Hence we have $5 \xrightarrow{d} 2$, $2 \xrightarrow{a} 3$, $2 \xrightarrow{a} 4$ and $2 \xrightarrow{a} 6$.

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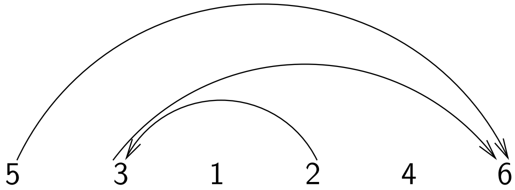

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Similarly we have $p(3) = 6$ and so the only ascent starting at 3 is $3 \xrightarrow{a} 6$. The parent of the numbers $\pi(3) = 1$, $\pi(5) = 4$ and $\pi(6) = 6$ is ∞, no arc begins at these vertices, no ascent starts at these vertices.

Existence and non-uniqueness

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For any alt-acyclic tournament T, the element 1 is always incomparable to the other elements of $\{1, \ldots, n\}$ in the right alternating walk order, hence the partial order $\leq_{ra}$ has always at least two linear extensions. This makes the use of biordered forests to directly count alt-acyclic tournaments difficult.

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Just avoiding inessential dimensions.

We associate to each region R of $\mathcal{H}_{2n-2}$ a tournament $T(R)$ on $\{1, \ldots, n\}$ as follows. For each $i < j$, set $i \to j$ iff the points of the region satisfy $x_i - y_i > x_j$.

Theorem

The correspondence $R \mapsto T(R)$ establishes a bijection between all regions of the homogenized Linial arrangement $\mathcal{H}_{2n-2}$ and all alternation acyclic tournaments on the set $\{1, \ldots, n\}$

The key to the proof is to set

$$x_i = \frac{n+1}{2} - \pi^{-1}(i) \quad \text{for } i = 1, 2, \ldots, n$$

and $y_i := \pi^{-1}(p(i)) - \pi^{-1}(i) - 1/2$ for $i = 1, \ldots, n-1$.

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The difference $x_i - x_j = \pi^{-1}(j) - \pi^{-1}(i)$ is the difference between the positions of j and i. This integer is strictly more than $y_i = \pi^{-1}(p(i)) - \pi^{-1}(i) - 1/2$ exactly when $j = p(i)$ or j is to the right of $p(i)$ in π. Therefore $T(R)$ is exactly the tournament induced by the biordered forest whose code is (π, p).

Interlude: counting regions in a hyperplane arrangement

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where $\chi(\mathcal{A}, q)$ is the *characteristic polynomial* of the arrangement. We may compute this, using *Athanasiadis' formula*. We consider the hyperplanes defined by the same equations in a vector space of the same dimension over a finite field $\mathbb{F}_q$ with q elements, where q is a large prime number. $\chi(\mathcal{A}, q)$ is then the number of points in the vector space that do not belong to any hyperplane:

$$\chi(\mathcal{A}, q) = \left| \mathbb{F}_q^d - \bigcup \mathcal{A} \right|.$$

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$$\chi(n,k,q) = (q-k)\cdot k\cdot\chi(n-1,k,q)+(q-k+1)^2\cdot\chi(n-1,k-1,q)$$

for $n \geq 2$, and the initial condition $\chi(1,k,q) = \delta_{1,k}q^2$.

Two helpful miracles

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Substituting $q = -1$, we realize that the numbers $\frac{(-1)^{n-k} \cdot \chi(n,k,-1)}{(k!)^2}$ satisfy the same recurrence and initial conditions as the one found by Andrews, Gawronski and Littlejohn for the *Legendre-Stirling numbers*.

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The Genocchi numbers

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The *Genocchi numbers* G_n *of the first kind* are given by the exponential generating function

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The unsigned Genocchi number $|G_{2n+2}|$ is the number of excedant functions $f : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ satisfying $f(\{1, \ldots, 2n\}) = \{2, 4, \ldots, 2n\}$.

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A function is *excedant* if $f(i) \geq i$ holds for all i.

The Genocchi numbers

Equivalently

Corollary

The unsigned Genocchi number $|G_{2n+2}|$ is the number of ordered pairs

$$((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \in \mathbb{Z}^n \times \mathbb{Z}^n$$

such that $1 \leq a_i, b_i \leq i$ hold for all i and the set $\{a_1, b_1, \ldots, a_n, b_n\}$ equals $\{1, \ldots, n\}$.

The median Genocchi numbers

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Using the largest maximum order

Using the largest maximum order

For an alternation acyclic tournament T on $\{1,\ldots,n\}$, we define the *largest maximal order* to be the permutation $\lambda = \lambda(1)\cdots\lambda(n)$, in which for each k, the vertex $\lambda(k)$ is the largest maximal element in the poset obtained by restricting the partial order $\leq_{ra}$ to the set $\{\lambda(1),\ldots,\lambda(k)\}$. We call the unique pair (λ, p) inducing T the *largest maximal representation of* T.

Using the largest maximum order

Theorem

Given a permutation λ of $\{1, \ldots, n\}$ and a parent function

$$p : \{1, 2, \ldots, n\} \to \{2, \ldots, n\} \cup \{\infty\},$$

the pair (λ, p) is the largest maximal representation of the tournament induced by (λ, p) if and only if for each descent i of λ, the vertex $\lambda(i+1)$ belongs to the range of p.

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Recursive counting

Recursive counting

We say that an alternation acyclic tournament has *type* (n,i,j) if it is a tournament on $\{1,\ldots,n\}$, and the parent function p in its largest maximal representation (λ,p) satisfies $|p^{-1}(\infty)|=i$ and $|p(\{1,\ldots,n\})|=j+1$. We will denote the number of alternation acyclic tournaments of type (n,i,j) with $A(n,i,j)$.

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Theorem

The numbers $A(n, i, j)/j!$ are integers, given by $A(1, i, j)/j! = \delta_{i,1} \cdot \delta_{0,j}$ and the recurrence

$$\frac{A(n,i,j)}{j!} = \sum_{k=i}^{n-1} \binom{k}{i-1} \cdot \frac{A(n-1,k,j-1)}{(j-1)!} + (j+1) \cdot \frac{A(n-1,i-1,j)}{j!}$$

for $n \geq 2$.

Sample tables

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For $n = 2$ the table for $A(2, i, j)/j!$ is

1	1	
0	0	1
j \ i	1	2

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1	1	
0	0	1

For $n = 3$ the table for $A(3, i, j)/j!$ is

j \ i	1	2	3
2	1		
1	1	4	
0	0	0	1

Sample tables

For $n = 4$ the table for $A(4, i, j)/j!$ is

j \ i	1	2	3	4
3	1			
2	5	11		
1	1	5	11	
0	0	0	0	1

Sample tables

For $n = 5$ the table for $A(5, i, j)/j!$ is

j \ i					
4	1				
3	16	26			
2	17	58	66		
1	1	6	16	26	
0	0	0	0	0	1
	1	2	3	4	5

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1	1	6	16	26	
0	0	0	0	0	1

In the main diagonal of each table we have the *Eulerian* numbers: $A(n, n-j, j)/j!$ is the number of permutations of $\{1, \ldots, n\}$ having exactly j descents. (Easy.)

Sample tables

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j \ i	1	2	3	4	5
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0	0	0	0	0	1

The first column gives rise to the Genocchi numbers of the first kind.

Refined counting

Refined counting

Theorem

For each $i \in \{1, \ldots, n\}$, *the sum* $\sum_{j=0}^{n} A(n,i,j)$ *is the number of ordered pairs*

$$((a_1, \ldots, a_{n-1}), (b_1, \ldots, b_{n-1})) \in \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1}$$

satisfying the following conditions:

1. $0 \leq a_k \leq k$ *and* $1 \leq b_k \leq k$ *hold for all* $k \in \{1, \ldots, n-1\}$;
2. *the set* $\{a_1, b_1, \ldots, a_{n-1}, b_{n-1}\}$ *contains* $\{1, \ldots, n-1\}$;
3. $|\{k \in \{1, \ldots, n-1\} \ : \ a_k = 0\}| = i - 1$.

Refined counting

The key ingredient to proving the theorem is the following bijection.

Theorem

There is a bijection between the set of all permutations π *of* $\{1, \ldots, n\}$ *and the set of excedant functions* $f : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ *such that, for each* π*, a number* $k \in \{1, \ldots, n\}$ *does not belong to the set* $\{f(1), \ldots, f(n)\}$ *if and only if* $\pi(i+1) = k$ *for some descent* i *of* π*.*

Ascending alt-acyclic tournaments

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We call an alternation acyclic tournament T on $\{1, \ldots, n\}$ *ascending* if every $i < n$ is the tail of an ascent, that is, for each $i < n$ there is a $j > i$ such that $i \to j$.

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Lemma

An alternating acyclic tournament T on $\{1, \ldots, n\}$ is ascending if and only if it has type $(n, 1, j)$ for some j.

Ascending alt-acyclic tournaments

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Lemma

An alternating acyclic tournament T on $\{1, \dots, n\}$ is ascending if and only if it has type $(n, 1, j)$ for some j.

Corollary

The number of ascending alternation acyclic tournaments on $\{1, \dots, n\}$ is the unsigned Genocchi number of the first kind $|G_{2n}|$.

A new model for the median Genocchi numbers

A new model for the median Genocchi numbers

Corollary

The median Genocchi number H_{2n-1} is the total number of all ordered pairs

$$((a_1, \ldots, a_{n-1}), (b_1, \ldots, b_{n-1})) \in \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1}$$

such that $0 \leq a_k \leq k$ and $1 \leq b_k \leq k$ hold for all k and the set $\{a_1, b_1, \ldots, a_{n-1}, b_{n-1}\}$ contains $\{1, \ldots, n-1\}$.

A new model for the median Genocchi numbers

Theorem

The normalized median Genocchi number h_n *is the number of sequences* $\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_n, v_n\}$ *subject to the following conditions:*

1. *the set* $\{u_k, v_k\}$ *is a (one- or two-element) subset of* $\{1, \ldots, k\}$*;*
2. *the set* $\{u_1, v_1, u_2, v_2, \ldots, u_n, v_n\}$ *equals* $\{1, \ldots, n\}$.

A new model for the median Genocchi numbers

The key idea is the $\mathbb{Z}_2^n$-action:

$$(a'_k, b'_k) = \begin{cases} (b_k, a_k) & \text{if } a_k \neq b_k \text{ and } a_k \neq 0; \\ (0, b_k) & \text{if } a_k = b_k; \\ (b_k, b_k) & \text{if } a_k = 0. \end{cases}$$

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