A colored version of Brylawski's tensor product formula and its applications

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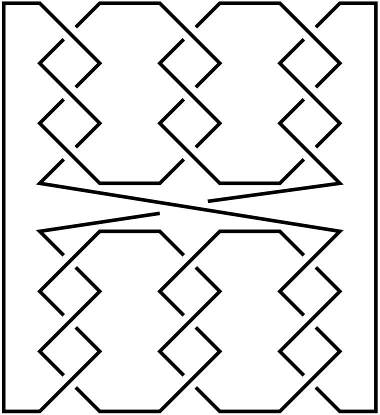
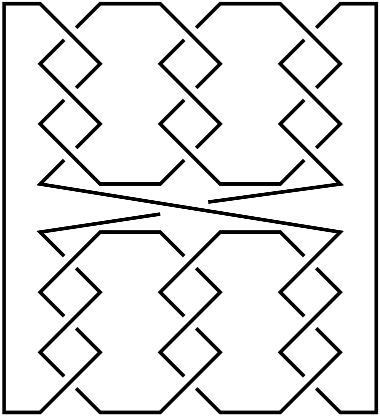
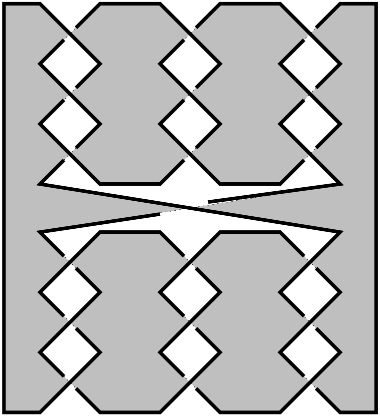


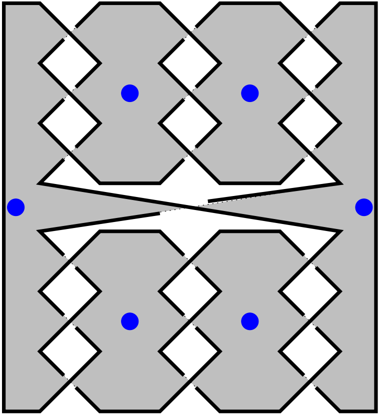
How to compute the Jones polynomial of this knot?



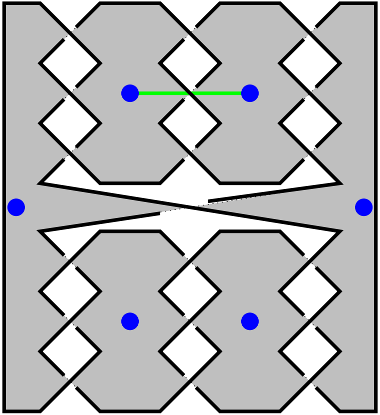
Draw the knot in the plane.



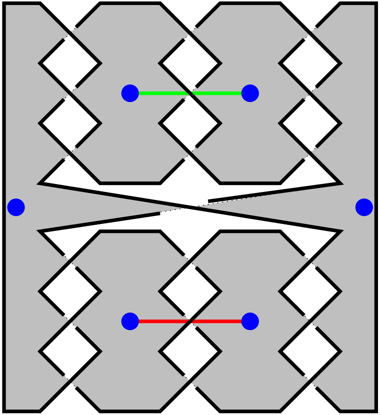
Two-color its regions.



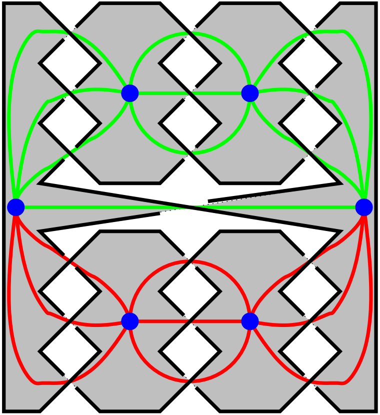
Put a vertex in the middle of each dark region.



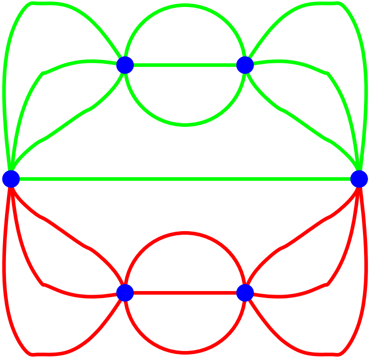
Draw a positive edge across each positive crossing.



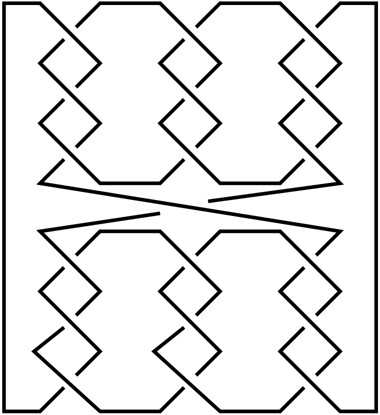
Draw a negative edge across each negative crossing.



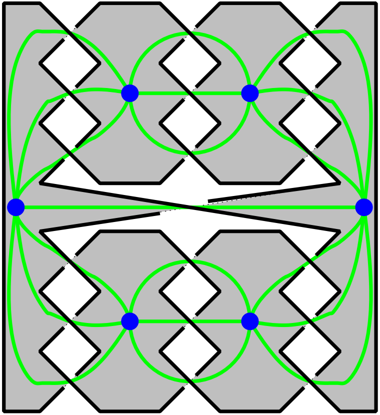
Obtain a signed graph.



Compute the *signed Tutte polynomial* of this signed graph . . .



A similar *alternating* knot, yielding only positive edges.



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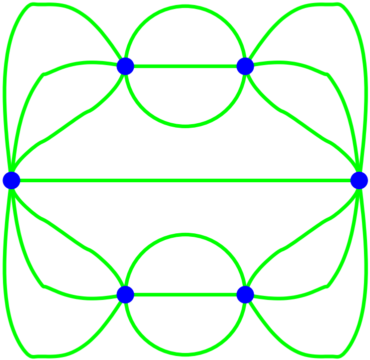

A similar *alternating* knot, yielding only positive edges.

Definition

The *signed Tutte polynomial* $T(G; A_+, A_-, B_+, B_-, x_+, x_-, y_+, y_-)$ of a graph is given recursively by $T(.) = 1$ and

$$T(G) = \begin{cases} x_\varepsilon T(G) & \text{if } e \text{ is a coloop;} \\ y_\varepsilon T(G) & \text{if } e \text{ is a loop;} \\ A_\varepsilon T(G/e) + B_\varepsilon T(G \setminus e) & \text{otherwise} \end{cases}$$

Here ε is the sign of the edge e.

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Here ε is the sign of the edge e.

Setting $x_\varepsilon = x$, $y_\varepsilon = y$, $A_\varepsilon = 1$ and $B_\varepsilon = 1$ yields the original definition of the Tutte polynomial.

The *Kauffman bracket* is given by

$$\langle D\rangle = T(G(D); A, A^{-1}, A^{-1}, A, -A^{-3}, -A^{3}, -A^{3}, -A^{-3}).$$

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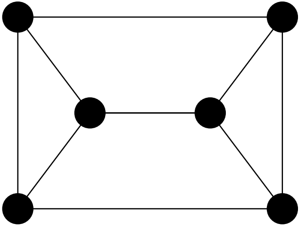
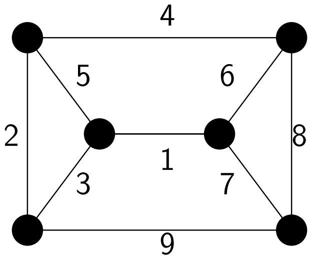
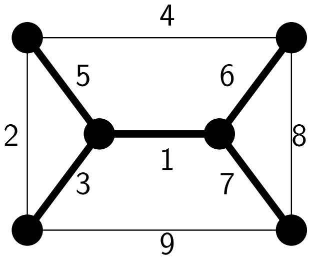
If we substitute $A^4 = t^{-1}$ in $(-A^{-3})^{w(D)}\langle D\rangle$, then we obtain the Jones polynomial of the knot D . Here $w(D)$ is the *writhe* of the knot.



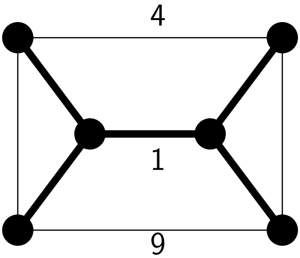
Consider a connected graph.



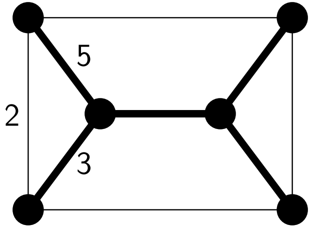
Number its edges.



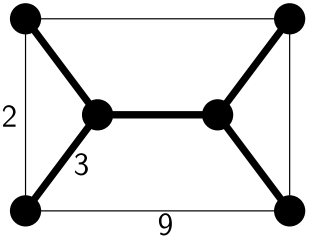
With respect to each spanning tree, each edge is internally or externally active or inactive.



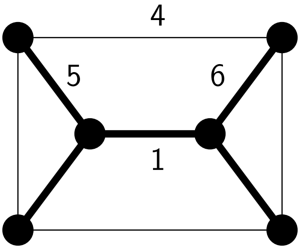
1 is internally active because all external edges in the unique cocycle closed by 1 have a larger number.



2 is externally active because all internal edges in the unique cycle closed by 2 have a larger number.



3 is internally inactive because the external edge 2 in the unique cocycle closed by 3 is larger.



4 is externally inactive because the internal edge 1 in the unique cycle closed by 4 is larger.

Theorem (Tutte)

The Tutte polynomial $T(G; x, y)$ of a connected graph G is the total weight of all spanning trees of G, where the weight of each spanning tree is the product of the weights of the edges with respect to this spanning tree: internally active edges have weight x, externally active edges have weight y, all other edges have weight 1.

internally active	X_λ	externally active	Y_λ
internally inactive	x_λ	externally inactive	y_λ

Table: Variable assignment for an edge of color λ.

Definition

Number the edges of a connected colored graph G, and define the weight of each edge with respect to each spanning tree using the table above. Define the colored Tutte polynomial $T(G)$ as the total weight of all spanning trees.

Theorem (Bollobás and Riordan)

The colored Tutte polynomial, defined as above, is independent of the labeling if and only if we factor $\mathbb{Z}[\Lambda] := \mathbb{Z}[x_\lambda, y_\lambda, X_\lambda, Y_\lambda \ : \ \lambda \in \Lambda]$ *by an ideal I such that the differences* $\det\begin{pmatrix} X_\lambda & y_\lambda \\ X_\mu & y_\mu \end{pmatrix} - \det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix}$, $Y_\nu \det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} - Y_\nu \det\begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix}$ *and* $X_\nu \det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} - X_\nu \det\begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix}$ *belong to I.*

Remark

In our examples the values assigned to the variables x_λ, y_λ, X_λ and Y_λ are not zero. The ideal generated by all polynomials of the forms $\det\begin{pmatrix} X_\lambda & y_\lambda \\ X_\mu & y_\mu \end{pmatrix} - \det\begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix}$ *and* $\det\begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix} - \det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix}$ *is a prime ideal.*

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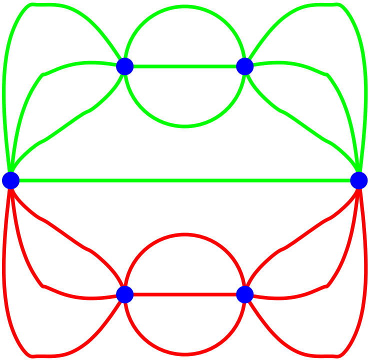
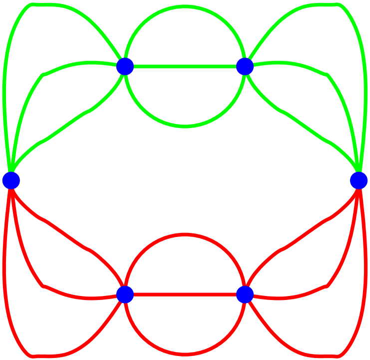
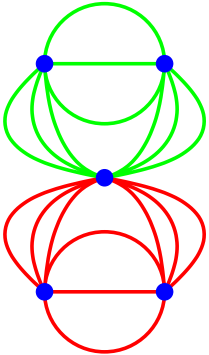
We consider the colored Tutte polynomial as an element of $\mathbb{Z}[\Lambda]/I_1$, where I_1 is the prime ideal generated by the above differences of determinants.



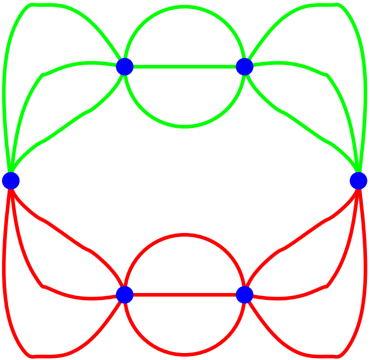
Let us return to the signed graph of our "motivating example".



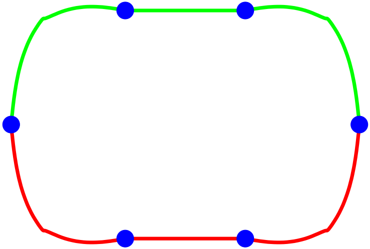
Deleting the horizontal edge in the middle gives this graph.



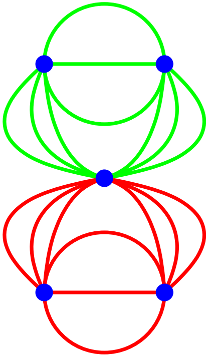
Contracting the horizontal edge in the middle gives this graph.



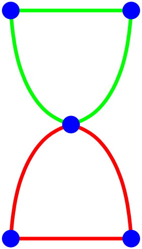
The graph obtained by deleting the middle edge is also obtained by "triplicating" each edge . . .



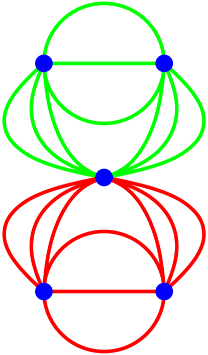
. . . in this graph.



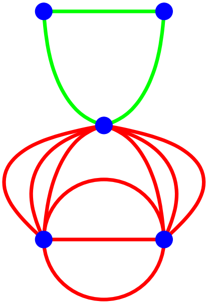
The graph obtained by contracting the middle edge is also obtained by "triplicating" each edge ...



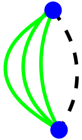
... in this graph.



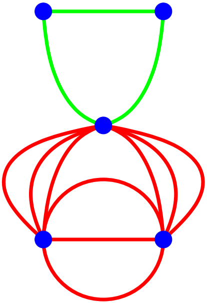
We will say that this graph is the "green" tensor product of . . .



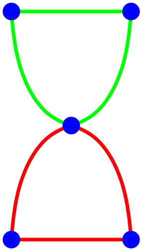
. . . this graph, and of . . .



…this graph.



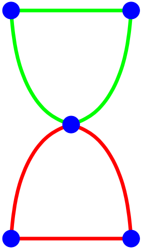
Similarly, this graph is the "red" tensor product of . . .



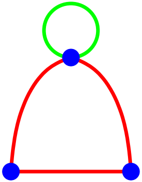
. . . this graph, and of . . .

...this graph.

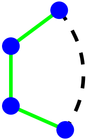
NOT OVER YET!



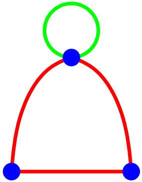
This graph is the "green" tensor product of ...



...this graph, and of ...



...this graph, and



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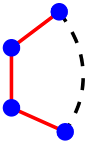

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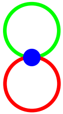

. . . this graph.

Definition

Let M and N be two graphs colored with the set Λ, $\lambda \in \Lambda$ a fixed color, and e a distinguished edge of N that is neither a loop nor a bridge. The λ-tensor product of M and N, denoted by $M \otimes_\lambda N$ is the colored graph obtained by replacing each edge in M of color λ with a copy of $N \setminus e$, where the distinguished edge e is to be identified with the replaced edge of M.

Remark

When $|\Lambda| = 1$, i.e., the graph is not colored, we obtain Brylawski's definition of a tensor product of two matroids, specialized to graphs.

Theorem (Brylawski)

The Tutte polynomial $T(M \otimes N_e) \in \mathbb{Z}[x,y]$ may be obtained from $T(M) \in \mathbb{Z}[x,y]$ by substituting $T(N \setminus e)/T_L(N,e)$ into x, $T(N/e)/T_C(N,e)$ into y, and multiplying the resulting rational expression with $T_L(N,e)^{r(M)} T_C(N,e)^{|M|-r(M)}$. That is,

$$T(M \otimes N_e) =$$

$$T_L(N,e)^{r(M)} T_C(N,e)^{|M|-r(M)} \cdot T\left(M; \frac{T(N \setminus e)}{T_L(N,e)}, \frac{T(N/e)}{T_C(N,e)}\right).$$

Here $T_L(N,e)$ are defined by the system of equations

$$\begin{aligned} T(N/e) - T_C(N,e) &= (y-1)T_L(N,e) \\ T(N \setminus e) - T_L(N,e) &= (x-1)T_C(N,e). \end{aligned}$$

Brylawski's formula was used to prove the following result.

Theorem (Jaeger–Vertigan–Welsh)

To compute the Jones polynomial of an alternating knot is $\#P$-hard.

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Theorem (Jaeger–Vertigan–Welsh)

To compute the Jones polynomial of an alternating knot is #P-hard.

To compute the Jones polynomial of an alternating knot, one only needs to know the (unsigned) Tutte polynomial of the associated graph.

Theorem (Diao-H.-Hinson)

Let M be a colored graph and N a colored graph with a distinguished edge e that is neither a loop nor a bridge. Then the ordinary Tutte polynomial $T(M \otimes_\lambda N)$ can be computed from $T(M)$ by keeping all variables of color $\mu \neq \lambda$ unchanged, and using the substitutions $X_\lambda \mapsto T(N \setminus e)$, $x_\lambda \mapsto T_L(N, e)$, $Y_\lambda \mapsto T(N/e)$ and $y_\lambda \mapsto T_C(N, e)$.

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But what are $T_C(N, e)$ and $T_L(N, e)$?

Definition

Define $T_L(N, e)$ by the same rule as $T(N \setminus e)$ except that internally active edges on a cycle closed by e will be considered as internally inactive instead.
Define $T_C(N, e)$ by the same rule as $T(N/e)$ except that externally active edges that would close a cycle containing e will be considered as externally inactive instead.

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Motto: "e has the smallest label."

Theorem (Diao-H.-Hinson)

The following two equalities hold:

$$x_\lambda(T(N/e) - T_C(N,e)) = (Y_\lambda - y_\lambda)T_L(N,e), \qquad (1)$$

$$y_\lambda(T(N \setminus e) - T_L(N,e)) = (X_\lambda - x_\lambda)T_C(N,e). \qquad (2)$$

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Corollary (Diao-H.-Hinson)

The polynomials $T_C(N,e)$ and $T_L(N,e)$ are independent of the labeling. *They may be equivalently defined by all equations (1) and (2).*

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The polynomials $T_C(N,e)$ and $T_L(N,e)$ are independent of the labeling. *They may be equivalently defined by all equations (1) and (2).*

Here we use that we have factored by a *prime ideal*.

Equations (1) and (2) are also equivalent to:

$$\det\begin{pmatrix} T_L(N,e) & T_C(N,e) \\ x_\lambda & y_\lambda \end{pmatrix} = \det\begin{pmatrix} T_L(N,e) & T(N/e) \\ x_\lambda & Y_\lambda \end{pmatrix} \quad (3)$$

and

$$\det\begin{pmatrix} T_L(N,e) & T_C(N,e) \\ x_\lambda & y_\lambda \end{pmatrix} = \det\begin{pmatrix} T(N\setminus e) & T_C(N,e) \\ X_\lambda & y_\lambda \end{pmatrix}. \quad (4)$$

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This reformulation implies that the substitutions $X_\lambda \mapsto T(N\setminus e)$, $x_\lambda \mapsto T_L(N,e)$, $Y_\lambda \mapsto T(N/e)$ and $y_\lambda \mapsto T_C(N,e)$ induce an endomorphism of $\mathbb{Z}[\Lambda]/I_1$.

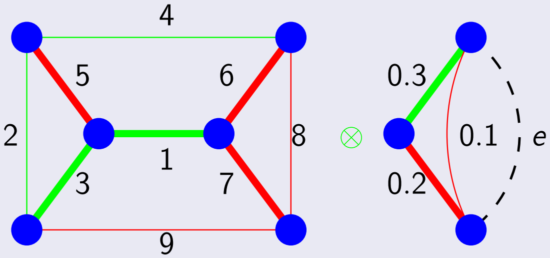
The proof of (1) and (2) uses some nontrivial combinatorics and the fact that the following identities hold in $\mathbb{Z}[\Lambda]/I_1$:

$$x_\lambda \left(\prod_{i=1}^{k} Y_{\lambda_i} - \prod_{i=1}^{k} y_{\lambda_i} \right) = (Y_\lambda - y_\lambda) \sum_{i=1}^{k} x_{\lambda_i} \prod_{j=1}^{i-1} Y_{\lambda_j} \prod_{t=i+1}^{k} y_{\lambda_t},$$

$$y_\lambda \left(\prod_{i=1}^{k} X_{\lambda_i} - \prod_{i=1}^{k} x_{\lambda_i} \right) = (X_\lambda - x_\lambda) \sum_{i=1}^{k} y_{\lambda_i} \prod_{j=1}^{i-1} X_{\lambda_j} \prod_{t=i+1}^{k} x_{\lambda_t}.$$

Proof of the colored tensor product formula.

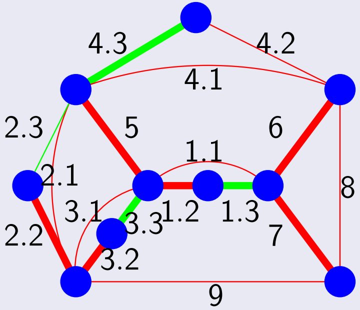
A spanning tree of



□

Proof of the colored tensor product formula.

is



□

The Jones polynomial of our motivating example is

$$V_K(t) = t^{-10}(1 - 4t + 12t^2 - 26t^3 + 49t^4 - 74t^5 + 96t^6 - 112t^7 + 110t^8 - 97t^9 + 77t^{10} - 47t^{11} + 23t^{12} - 8t^{13} - 2t^{14} + 3t^{15} - t^{16} + t^{17}).$$

Matches the result found by the program Knotscape.
For the Kauffman brackets, the homomorphic images of $T_C(N)$ and $T_L(N)$ are the solutions of the system of equations

$$\begin{array}{rcl} (-A^3 - A^{-1}) \cdot z_L + A \cdot z_C & = & A \cdot \langle N/e \rangle \\ A^{-1} \cdot z_L + (-A^{-3} - A) \cdot z_C & = & A^{-1} \cdot \langle N \setminus e \rangle. \end{array} \qquad (5)$$

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Swept under the rug: We need to consider the disconnected graph generalization of the colored Tutte polynomial.

The pointed random-cluster-generating functions $Z_C(N, e; p, \kappa)$ and
$Z_L(N, e; p, \kappa)$ are given by

$$\begin{aligned} Z_C(N, e; p, \kappa) &= \frac{Z(N \setminus e; p, \kappa) - Z(N/e; p, \kappa)}{\kappa - 1}, \\ Z_L(N, e; p, \kappa) &= \frac{\kappa Z(N/e; p, \kappa) - Z(N \setminus e; p, \kappa)}{\kappa - 1}. \end{aligned}$$

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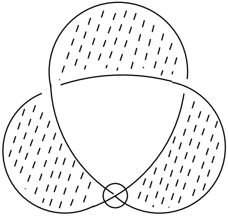
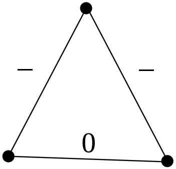
Proposition (Diao-H.-Hinson)

The probability that the endpoints of e become disconnected after an accident in $N \setminus e$ is $Z_C(N, e; p, 1)$, and the probability that they remain connected is $Z_L(N, e; p, 1)$.





Kauffman has a theory of *virtual knots* for knots drawn on different surfaces. These may be drawn in the plane with *virtual crossings*. There is an alternative approach (Chmutov, Pak, Kamada) , using the *Bollobás-Riordan polynomial* (unrelated to the colored Tutte polynomial). Chmutov has established a link between the two approaches.

Let G be a graph and $\mathcal{H} \subseteq E(G)$.
$\mathcal{C} \subseteq E(G) \setminus \mathcal{H}$ is a *contracting set* if it contains no cycles and $\mathcal{D} = E(G) \setminus (\mathcal{C} \cup \mathcal{H})$ is the corresponding *deleting set*).
Label the edges ($\phi : E(G) \to \mathbb{R}_+$) in $\mathcal{H}$ with 0 and the edges in $E(G) \setminus \mathcal{H}$ with distinct positive integers.

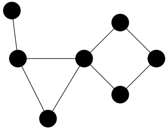
a) an edge $e \in \mathcal{C}$ is *internally active* if $\mathcal{D} \cup \{e\}$ contains a cocycle D_0 in which e is the smallest edge. otherwise it is *internally inactive*.
b) an edge $f \in \mathcal{D}$ is called *externally active* if $\mathcal{C} \cup \{f\}$ contains a cycle C_0 in which f is the smallest edge.

Let ψ be a mapping defined on the isomorphism classes of finite connected graphs with values in a ring $\mathcal{R}$. Assume ψ is a *block invariant*, i.e., for any connected graph G having n blocks $G_1, \ldots,$ G_n we have

$$\psi(G) = f_n(\psi(G_1), \ldots, \psi(G_n)),$$

for some $f_n : \mathcal{R}^n \to \mathcal{R}$ that is symmetric under permuting its input variables.

Assume also that ψ is invariant under *vertex pivots*:



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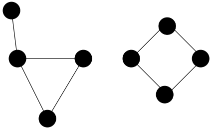

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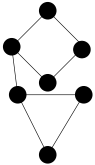

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We define the *relative Tutte polynomial* as

$$T^{\psi}_{\mathcal{H}}(G) = \sum_{\mathcal{C}} \Big(\prod_{e \in G \setminus H} w(G, c, \phi, \mathcal{C}, e)\Big)\psi(\mathcal{H}_{\mathcal{C}}) \in \mathcal{R}[\Lambda]. \qquad (6)$$

We have the following analogue of the Bollobás-Riordan theorem:

Theorem (Diao-H.)

Assume I is an ideal of $\mathcal{R}[\Lambda]$. Then the homomorphic image of $T_{\mathcal{H}}(G,\phi)$ in $\mathcal{R}[\Lambda]/I$ is independent of ϕ (for any G and ψ) if and only if

$$\det\begin{pmatrix} X_\lambda & y_\lambda \\ X_\mu & y_\mu \end{pmatrix} - \det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} \in I \tag{7}$$

and

$$\det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} - \det\begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix} \in I. \tag{8}$$

hold for all $\lambda, \mu \in \Lambda$.

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Please read:

[1] Tutte Polynomials of Tensor Products of Signed Graphs and their Applications in Knot Theory, *Journal of Knot Theory and Its Ramifications* **18** (2009), 561–589. (With Y. Diao and K. Hinson.)

[2] A Tutte-style proof of Brylawski's tensor product formula, *European Journal of Combinatorics* **32** (2011), 775–781. (With Y. Diao and K. Hinson.)

[3] Invariants of composite networks arising as a tensor product, *Graphs and Combinatorics* **25** (2009), 273-290. (With Y. Diao and K. Hinson.)

[4] Relative Tutte Polynomials for Colored Graphs and Virtual Knot Theory, *Combinatorics, Probability & Computing*, **19** (2010), 343-369. (With Y. Diao.)