The dual of the type B permutohedron as a Tchebyshev triangulation

Gábor Hetyei

Department of Mathematics and Statistics
University of North Carolina at Charlotte
`http://webpages.uncc.edu/ghetyei/`

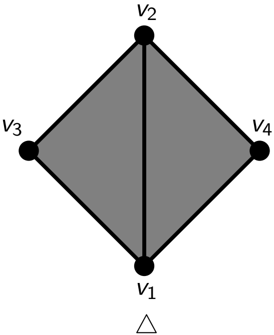
July 17, 2020, FPSAC online
arXiv:2007.07362 [math.CO]

Visual definition

Pull the midpoints of all edges in some order.

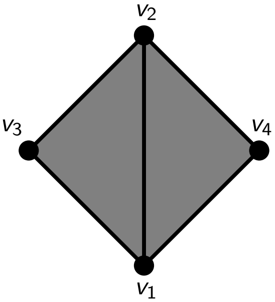
Visual definition

Pull the midpoints of all edges in some order.



Visual definition

Pull the midpoints of all edges in some order.

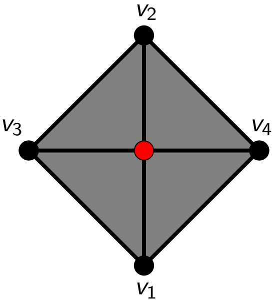


$\triangle$

$f_{-1} = 1,\ f_0 = 4,\ f_1 = 5,\ f_2 = 2.$

Visual definition

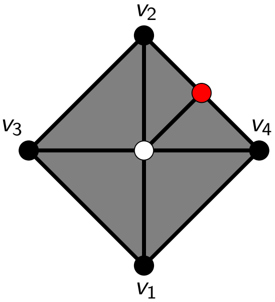
Pull the midpoints of all edges in some order.



$\triangle'$

Visual definition

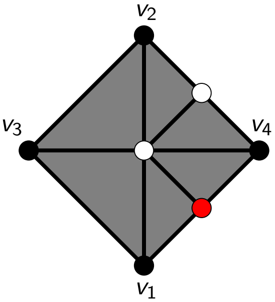
Pull the midpoints of all edges in some order.



$\triangle'$

Visual definition

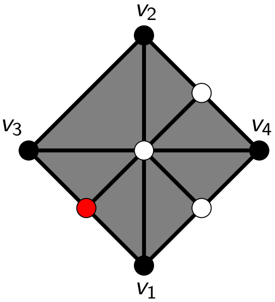
Pull the midpoints of all edges in some order.



$\triangle'$

Visual definition

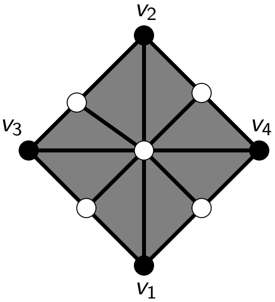
Pull the midpoints of all edges in some order.



$\triangle'$

Visual definition

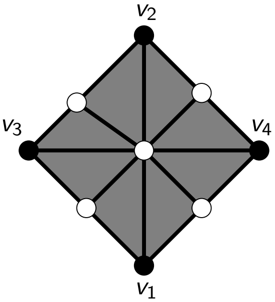
Pull the midpoints of all edges in some order.



$\triangle'$

Visual definition

Pull the midpoints of all edges in some order.

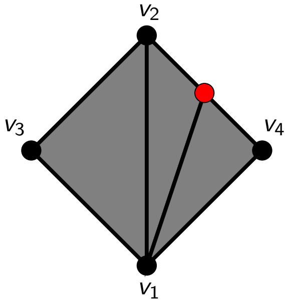


$\triangle'$

$f_{-1} = 1,\ f_0 = 9,\ f_1 = 16,\ f_2 = 8.$

Visual definition

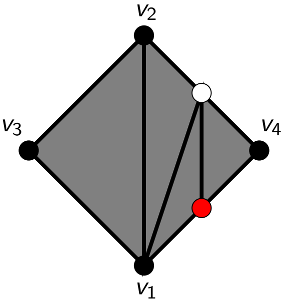
Pull the midpoints of all edges in some order.



$\triangle''$

Visual definition

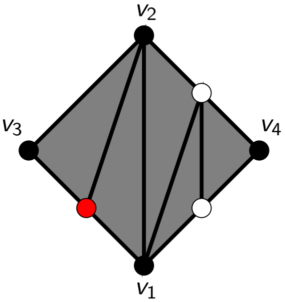
Pull the midpoints of all edges in some order.



$\triangle''$

Visual definition

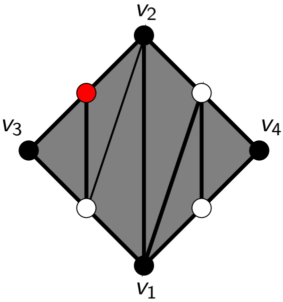
Pull the midpoints of all edges in some order.



$\triangle''$

Visual definition

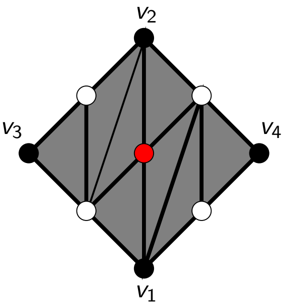
Pull the midpoints of all edges in some order.



$\triangle''$

Visual definition

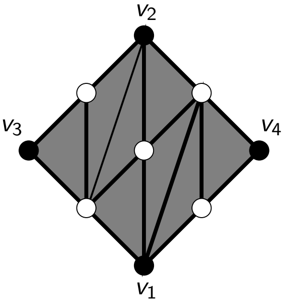
Pull the midpoints of all edges in some order.



$\triangle''$

Visual definition

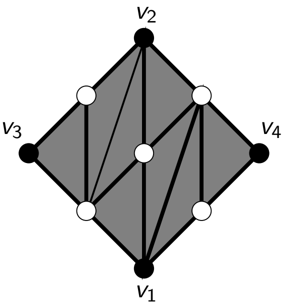
Pull the midpoints of all edges in some order.



$\triangle''$

Visual definition

Pull the midpoints of all edges in some order.



$\triangle''$

$f_{-1} = 1,\ f_0 = 9,\ f_1 = 16,\ f_2 = 8.$

Visual definition

Theorem (H.–Nevo)

All Tchebyshev triangulations of the same simplicial complex have the same face numbers.

Why the name Tchebyshev?

Why the name Tchebyshev?

Define the *F-polynomial* of a simplicial complex by
$F(\triangle) = \sum_{j=0}^{d} f_{j-1} \cdot \left(\frac{x-1}{2}\right)^{j}$.

Why the name Tchebyshev?

Define the *F-polynomial* of a simplicial complex by
$F(\triangle) = \sum_{j=0}^{d} f_{j-1} \cdot \left(\frac{x-1}{2}\right)^j$.
For our original complex

$$F(\triangle, x) = 1 + 4 \cdot \left(\frac{x-1}{2}\right) + 5 \cdot \left(\frac{x-1}{2}\right)^2 + 2 \cdot \left(\frac{x-1}{2}\right)^3$$
$$= \frac{x + 2x^2 + x^3}{4}$$

Why the name Tchebyshev?

Define the *F-polynomial* of a simplicial complex by $F(\triangle) = \sum_{j=0}^{d} f_{j-1} \cdot \left(\frac{x-1}{2}\right)^j$.

$$F(\triangle, x) = \frac{x + 2x^2 + x^3}{4}$$

Why the name Tchebyshev?

Define the *F-polynomial* of a simplicial complex by $F(\triangle) = \sum_{j=0}^{d} f_{j-1} \cdot \left(\frac{x-1}{2}\right)^{j}$.

$$F(\triangle, x) = \frac{x + 2x^2 + x^3}{4}$$

For a Tchebyshev triangulation

$$F(T(\triangle), x) = 1 + 9\left(\frac{x-1}{2}\right) + 16 \cdot \left(\frac{x-1}{2}\right)^2 + 8 \cdot \left(\frac{x-1}{2}\right)^3$$
$$= \frac{-1 - x + 2x^2 + x^3}{2}$$

Why the name Tchebyshev?

Define the *F-polynomial* of a simplicial complex by $F(\triangle) = \sum_{j=0}^{d} f_{j-1} \cdot \left(\frac{x-1}{2}\right)^j$.

$$F(\triangle, x) = \frac{x + 2x^2 + x^3}{4}$$

$$F(T(\triangle), x) = \frac{-1 - x + 2x^2 + x^3}{2}$$

Why the name Tchebyshev?

Define the *F-polynomial* of a simplicial complex by $F(\triangle) = \sum_{j=0}^{d} f_{j-1} \cdot \left(\frac{x-1}{2}\right)^j$.

$$F(\triangle, x) = \frac{x + 2x^2 + x^3}{4}$$

$$F(T(\triangle), x) = \frac{-1 - x + 2x^2 + x^3}{2}$$

$$\frac{-1 - x + 2x^2 + x^3}{2} = \frac{x + 2(2x^2 - 1) + (4x^3 - 3x)}{4}$$

Why the name Tchebyshev?

Define the *F-polynomial* of a simplicial complex by $F(\triangle) = \sum_{j=0}^{d} f_{j-1} \cdot \left(\frac{x-1}{2}\right)^j$.

$$F(\triangle, x) = \frac{x + 2x^2 + x^3}{4}$$

$$F(T(\triangle), x) = \frac{-1 - x + 2x^2 + x^3}{2}$$

$$\frac{-1 - x + 2x^2 + x^3}{2} = \frac{x + 2(2x^2 - 1) + (4x^3 - 3x)}{4}$$

$$F(T(\triangle), x) = T(F(\triangle, x)), \quad \text{where } T(x^n) = T_n(x) = \cos(n \cdot \arccos x).$$

Tchebyshev triangulations of the second kind

Tchebyshev triangulations of the second kind

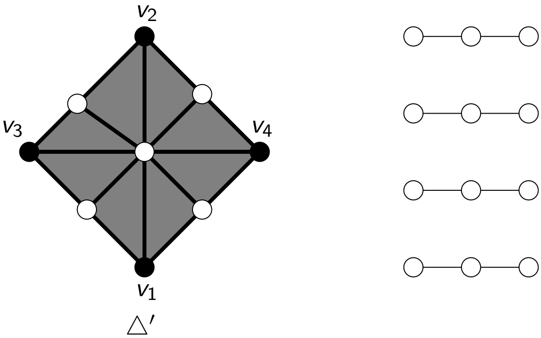
Defined as the multiset of links of the original vertices in a Tchebyshev triangulation.

Tchebyshev triangulations of the second kind

Defined as the multiset of links of the original vertices in a Tchebyshev triangulation.

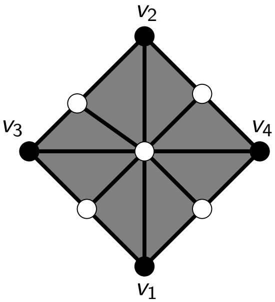
Tchebyshev triangulations of the second kind

Defined as the multiset of links of the original vertices in a Tchebyshev triangulation.

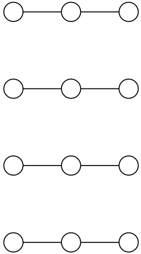


Tchebyshev triangulations of the second kind

Defined as the multiset of links of the original vertices in a Tchebyshev triangulation.



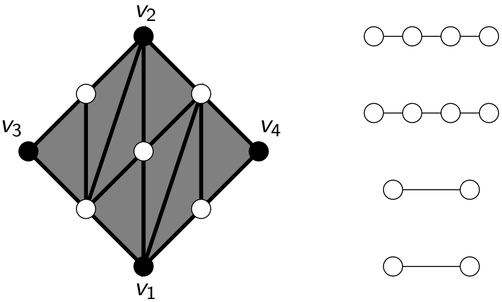
$\triangle'$



$f_{-1} = 4,\ f_0 = 12,\ f_1 = 8.$

Tchebyshev triangulations of the second kind

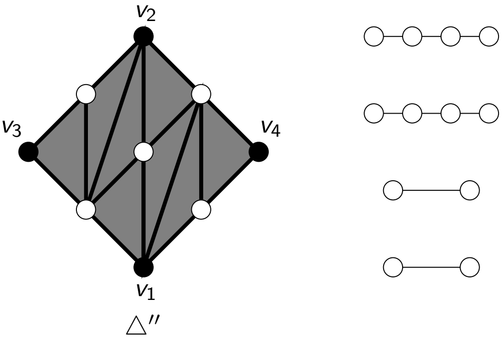
Defined as the multiset of links of the original vertices in a Tchebyshev triangulation.



$\triangle''$

Tchebyshev triangulations of the second kind

Defined as the multiset of links of the original vertices in a Tchebyshev triangulation.



$f_{-1} = 4, \ f_0 = 12, \ f_1 = 8.$

Tchebyshev triangulations of the second kind

Defined as the multiset of links of the original vertices in a Tchebyshev triangulation.

Theorem (H.–Nevo)

All Tchebyshev triangulations of the second kind the same simplicial complex have the same face numbers.

Why Tchebyshev triangulations of the second kind?

Why Tchebyshev triangulations of the second kind?

$$F(\triangle, x) = \frac{x + 2x^2 + x^3}{4}$$

Why Tchebyshev triangulations of the second kind?

$$F(\triangle, x) = \frac{x + 2x^2 + x^3}{4}$$

For a Tchebyshev triangulation of the second kind

$$\begin{aligned} F(U(\triangle), x) &= 4 + 12\left(\frac{x-1}{2}\right) + 8 \cdot \left(\frac{x-1}{2}\right)^2 \\ &= 2x^2 + 2x. \end{aligned}$$

Why Tchebyshev triangulations of the second kind?

$$F(\triangle, x) = \frac{x + 2x^2 + x^3}{4}$$

$$F(U(\triangle), x) = 2x^2 + 2x$$

Why Tchebyshev triangulations of the second kind?

$$F(\triangle, x) = \frac{x + 2x^2 + x^3}{4}$$

$$F(U(\triangle), x) = 2x^2 + 2x$$

$$2x^2 + 2x = 2 \cdot \frac{1 + 2 \cdot (2x) + (4x^2 - 1)}{4}$$

Why Tchebyshev triangulations of the second kind?

$$F(\triangle, x) = \frac{x + 2x^2 + x^3}{4}$$

$$F(U(\triangle), x) = 2x^2 + 2x$$

$$2x^2 + 2x = 2 \cdot \frac{1 + 2 \cdot (2x) + (4x^2 - 1)}{4}$$

$$\frac{1}{2} \cdot F(U(\triangle), x) = U(F(\triangle, x)), \quad \text{where } U(x^n) = U_{n-1}(x).$$

$$U_{n-1}(x) = \frac{\sin(n \cdot \arccos x)}{\sin(\arccos x))}.$$

The Tchebyshev transform $T(P)$ of a poset P

The Tchebyshev transform $T(P)$ of a poset P

The elements of $T(P)$ are the poset whose elements are the intervals $[x, y) \subset P$ satisfying $x \neq y$. We set $[x_1, y_1) \leq [x_2, y_2)$ if either $y_1 \leq x_2$ or both $x_1 = x_2$ and $y_1 \leq y_2$ hold.

The Tchebyshev transform $T(P)$ of a poset P

The elements of $T(P)$ are the poset whose elements are the intervals $[x, y) \subset P$ satisfying $x \neq y$. We set $[x_1, y_1) \leq [x_2, y_2)$ if either $y_1 \leq x_2$ or both $x_1 = x_2$ and $y_1 \leq y_2$ hold. Graded version (Ehrenborg-Readdy): given a graded poset P with minimum element $\widehat{0}$ and maximum element $\widehat{1}$, we introduce a new minimum element $\widehat{-1} < \widehat{0}$ and a new maximum element $\widehat{2}$. The *graded Tchebyshev transform of the first kind* of a graded poset P is then the interval $[[\widehat{-1}, \widehat{0}), [\widehat{1}, \widehat{2})]$ in $T(P \cup \{\widehat{-1}, \widehat{2}\})$.

The Tchebyshev transform $T(P)$ of a poset P

The elements of $T(P)$ are the poset whose elements are the intervals $[x, y) \subset P$ satisfying $x \neq y$. We set $[x_1, y_1) \leq [x_2, y_2)$ if either $y_1 \leq x_2$ or both $x_1 = x_2$ and $y_1 \leq y_2$ hold. Graded version (Ehrenborg-Readdy): given a graded poset P with minimum element $\widehat{0}$ and maximum element $\widehat{1}$, we introduce a new minimum element $\widehat{-1} < \widehat{0}$ and a new maximum element $\widehat{2}$. The *graded Tchebyshev transform of the first kind* of a graded poset P is then the interval $[[\widehat{-1}, \widehat{0}), [\widehat{1}, \widehat{2})]$ in $T(P \cup \{\widehat{-1}, \widehat{2}\})$.

Theorem

Then the order complex $\triangle(T(P) \setminus \{[\widehat{-1}, \widehat{0}), [\widehat{1}, \widehat{2})\})$ is a Tchebyshev triangulation of the suspension of $\triangle(P \setminus \{\widehat{0}, \widehat{1}\})$.

The Tchebyshev transform $T(P)$ of a poset P

The elements of $T(P)$ are the poset whose elements are the intervals $[x, y) \subset P$ satisfying $x \neq y$. We set $[x_1, y_1) \leq [x_2, y_2)$ if either $y_1 \leq x_2$ or both $x_1 = x_2$ and $y_1 \leq y_2$ hold. Graded version (Ehrenborg-Readdy): given a graded poset P with minimum element $\widehat{0}$ and maximum element $\widehat{1}$, we introduce a new minimum element $\widehat{-1} < \widehat{0}$ and a new maximum element $\widehat{2}$. The *graded Tchebyshev transform of the first kind* of a graded poset P is then the interval $[[\widehat{-1}, \widehat{0}), [\widehat{1}, \widehat{2})]$ in $T(P \cup \{\widehat{-1}, \widehat{2}\})$.

Theorem

Then the order complex $\triangle(T(P) \setminus \{[\widehat{-1}, \widehat{0}), [\widehat{1}, \widehat{2})\})$ *is a Tchebyshev triangulation of the suspension of* $\triangle(P \setminus \{\widehat{0}, \widehat{1}\})$.

For more information see the work of Ehrenborg and Readdy.

The poset of intervals inducing a Tchebyshev triangulation

The poset of intervals inducing a Tchebyshev triangulation

The *poset of intervals* $I(P)$ of a poset P is the poset of the intervals of P ordered by inclusion.

The poset of intervals inducing a Tchebyshev triangulation

The *poset of intervals* $I(P)$ of a poset P is the poset of the intervals of P ordered by inclusion.

Theorem (Walker)

The order complex of $I(P)$ *is identifiable with a triangulation of the order complex of* P.

The poset of intervals inducing a Tchebyshev triangulation

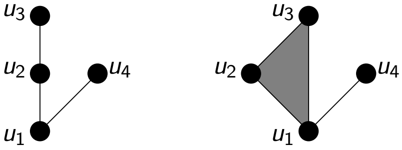
The *poset of intervals* $I(P)$ of a poset P is the poset of the intervals of P ordered by inclusion.

Theorem (Walker)

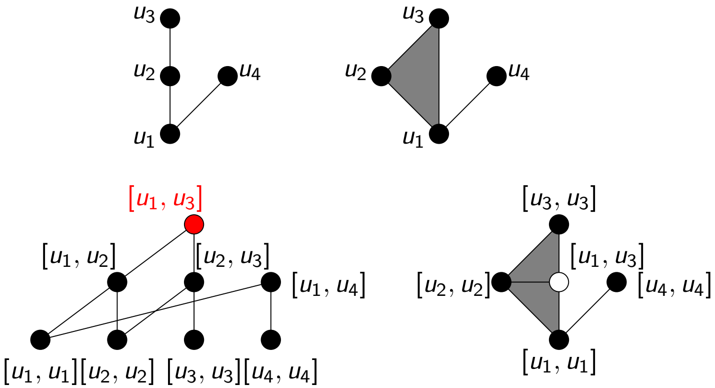
The order complex of $I(P)$ is identifiable with a triangulation of the order complex of P.

New proof: It is actually a Tchebyshev triangulation.

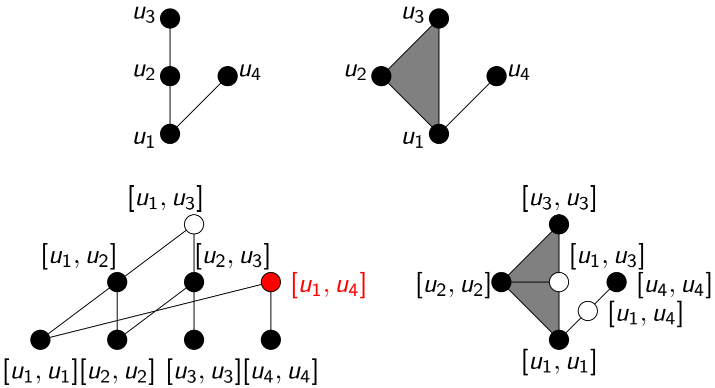
The poset of intervals inducing a Tchebyshev triangulation



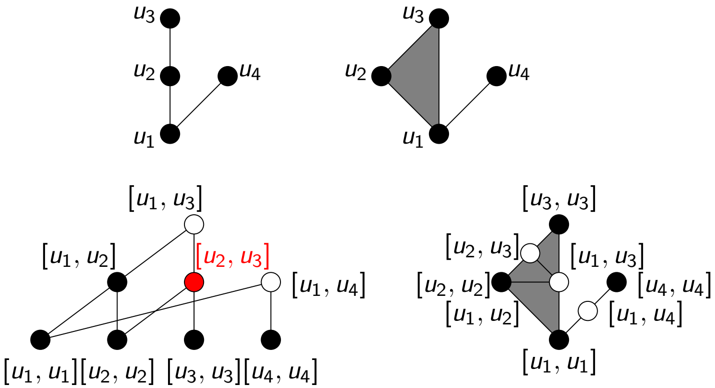
The poset of intervals inducing a Tchebyshev triangulation



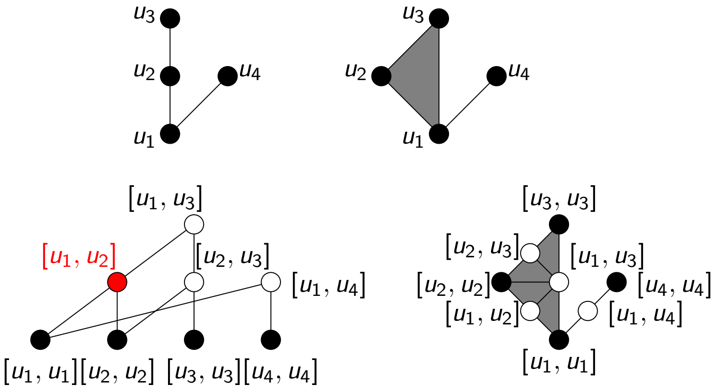
The poset of intervals inducing a Tchebyshev triangulation



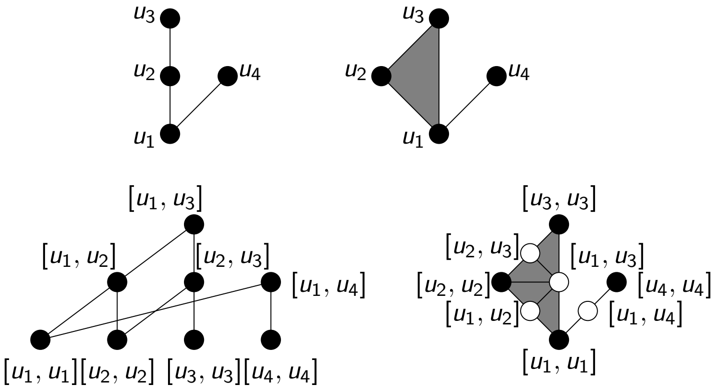
The poset of intervals inducing a Tchebyshev triangulation



The poset of intervals inducing a Tchebyshev triangulation



The poset of intervals inducing a Tchebyshev triangulation



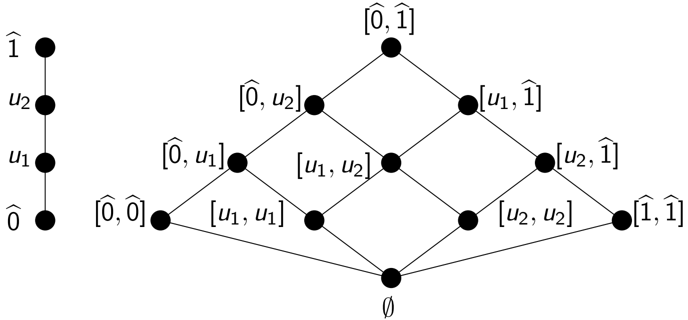
The graded poset $\widehat{I}(P)$ of intervals of a graded poset P

The graded poset $\widehat{I}(P)$ of intervals of a graded poset P

We just add $\emptyset$ as the unique minimum element.

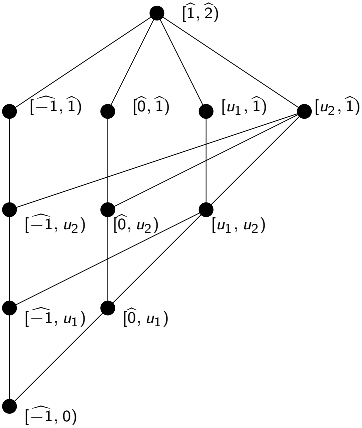
The graded poset $\widehat{I}(P)$ of intervals of a graded poset P

We just add $\emptyset$ as the unique minimum element.



The graded poset $\widehat{I}(P)$ of intervals of a graded poset P

Compare it with the Tchebyshev transform of a chain.



The graded poset $\widehat{I}(P)$ of intervals of a graded poset P

Proposition

The order complex $\triangle(\widehat{I}(P) - \{\emptyset, [\widehat{0}, \widehat{1}]\})$ *is a Tchebyshev triangulation of the suspension of* $\triangle(P - \{\widehat{0}, \widehat{1}\})$.

Known facts

Known facts

The dual of the type A permutohedron is the order complex of a Boolean algebra.

Known facts

Each facet of the n-dimensional type B permutohedron is uniquely labeled with a pair of sets (K^+, K^-) where K^+ and K^- is are subsets of $[1, n]$, satisfying $K^+ \subseteq [1, n] - K^-$ and K^+ and K^- cannot be both empty. For a set of valid labels

$$\{(K_1^+, K_1^-), (K_2^+, K_2^-), \ldots, (K_m^+, K_m^-)\}$$

the intersection of the corresponding set of facets is a nonempty face of $\mathrm{Perm}(B_n)$ if and only if

$$K_1^+ \subseteq K_2^+ \subseteq \cdots \subseteq K_m^+ \subseteq [1, n] - K_m^- \subseteq [1, n] - K_{m-1}^- \subseteq \cdots \subseteq [1, n] - K_1^-$$

Simple recoding

Simple recoding

Set $X := K^+$ and $Y := [1, n] - K^-$.

Simple recoding

Set $X := K^+$ and $Y := [1, n] - K^-$. The label of each facet becomes a nonempty interval $[X, Y]$ of the Boolean algebra of rank n that is different from $[\emptyset, [1, n]]$. The set $\{[X_1, Y_1], [X_2, Y_2], \ldots, [X_m, Y_m]\}$ labels a collection of facets with a nonempty intersection if and only if the intervals form an increasing chain in $\widehat{I}(P([1, n])) - \{\emptyset, [\emptyset, [1, n]]\}$.

Simple recoding

Set $X := K^+$ and $Y := [1, n] - K^-$. The label of each facet becomes a nonempty interval $[X, Y]$ of the Boolean algebra of rank n that is different from $[\emptyset, [1, n]]$. The set $\{[X_1, Y_1], [X_2, Y_2], \ldots, [X_m, Y_m]\}$ labels a collection of facets with a nonempty intersection if and only if the intervals form an increasing chain in $\widehat{I}(P([1, n])) - \{\emptyset, [\emptyset, [1, n]]\}$.

Simple recoding

Set $X := K^{+}$ and $Y := [1, n] - K^{-}$. The label of each facet becomes a nonempty interval $[X, Y]$ of the Boolean algebra of rank n that is different from $[\emptyset, [1, n]]$. The set $\{[X_1, Y_1], [X_2, Y_2], \ldots, [X_m, Y_m]\}$ labels a collection of facets with a nonempty intersection if and only if the intervals form an increasing chain in $\widehat{I}(P([1, n])) - \{\emptyset, [\emptyset, [1, n]]\}$.

Simple recoding

Set $X := K^{+}$ and $Y := [1, n] - K^{-}$. The label of each facet becomes a nonempty interval $[X, Y]$ of the Boolean algebra of rank n that is different from $[\emptyset, [1, n]]$. The set $\{[X_1, Y_1], [X_2, Y_2], \ldots, [X_m, Y_m]\}$ labels a collection of facets with a nonempty intersection if and only if the intervals form an increasing chain in $\widehat{I}(P([1, n])) - \{\emptyset, [\emptyset, [1, n]]\}$.

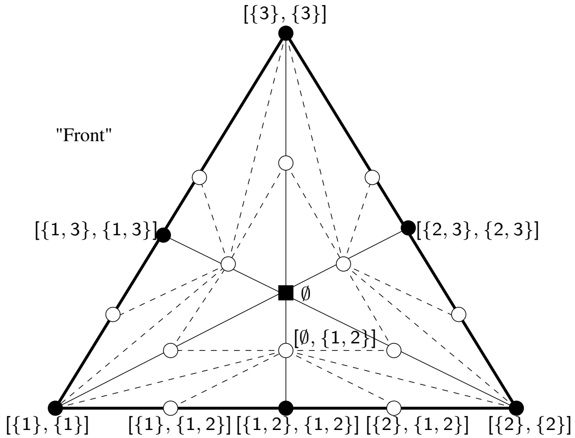
Proposition

The dual of $\mathrm{Perm}(B_n)$ *is a simplicial polytope whose boundary complex is combinatorially equivalent to a Tchebyshev triangulation of the suspension of* $\triangle(\widehat{I}(P([1, n])) - \{\emptyset, [\emptyset, [1, n]]\})$.

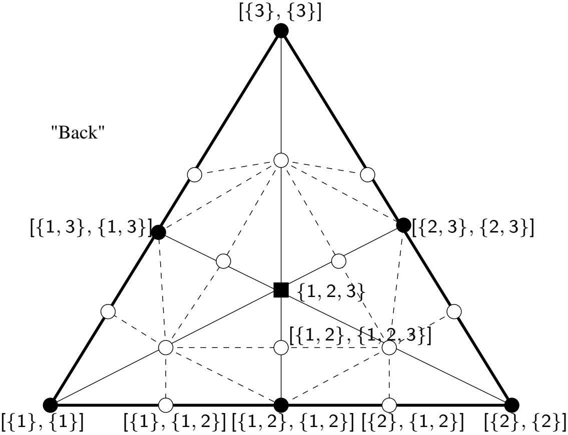
An illustration

An illustration

An illustration



An illustration



Looks familiar?

Looks familiar?

The poset of intervals of the Boolean algebra have been studied by:

Looks familiar?

The poset of intervals of the Boolean algebra have been studied by:

- Athanasiadis an Savvidou (type B derangement polynomials)

Looks familiar?

The poset of intervals of the Boolean algebra have been studied by:

- Athanasiadis an Savvidou (type B derangement polynomials)
- Anwar and Nazir (interval subdivisions)

Looks familiar?

The poset of intervals of the Boolean algebra have been studied by:

- Athanasiadis an Savvidou (type B derangement polynomials)
- Anwar and Nazir (interval subdivisions)

It is a consequence of the results of Anwar and Nazir that the h-polynomial of the type B Coxeter complex has real roots.

Looks familiar?

The poset of intervals of the Boolean algebra have been studied by:

- Athanasiadis an Savvidou (type B derangement polynomials)
- Anwar and Nazir (interval subdivisions)

It is a consequence of the results of Anwar and Nazir that the h-polynomial of the type B Coxeter complex has real roots. It is also a consequence of the real-rootedness of the derivative polynomials for the hyperbolic secant.

A new-old real-rootedness result

A new-old real-rootedness result

The F-polynomials of the type B Coxeter complexes have the same coefficients (up to sign) as the *derivative polynomials* $Q_n(x)$ *for secant*, defined by $\frac{d^n}{dx^n}\sec(x) = Q_n(\tan x)\cdot\sec(x)$.

A new-old real-rootedness result

The F-polynomials of the type B Coxeter complexes have the same coefficients (up to sign) as the *derivative polynomials $Q_n(x)$ for secant*, defined by $\frac{d^n}{dx^n}\sec(x) = Q_n(\tan x)\cdot\sec(x)$.

$$\sum_{j=0}^{n} f_{j-1}\left(\triangle\left(\widehat{I}(B_n) - \{\emptyset, \{1,\ldots,n\}\}\right)\right)\cdot\left(\frac{x-1}{2}\right)^j = \mathbf{i}^{-n}Q_n(x\cdot\mathbf{i}).$$

A new-old real-rootedness result

The F-polynomials of the type B Coxeter complexes have the same coefficients (up to sign) as the *derivative polynomials* $Q_n(x)$ *for secant*, defined by $\frac{d^n}{dx^n}\sec(x) = Q_n(\tan x) \cdot \sec(x)$.

Theorem

All roots of the derivative polynomials for hyperbolic tangent and secant are interlaced, real, and belong to the interval $[-1, 1]$.

A new-old real-rootedness result

The F-polynomials of the type B Coxeter complexes have the same coefficients (up to sign) as the *derivative polynomials $Q_n(x)$ for secant*, defined by $\frac{d^n}{dx^n}\sec(x) = Q_n(\tan x)\cdot\sec(x)$.

Theorem

All roots of the derivative polynomials for hyperbolic tangent and secant are interlaced, real, and belong to the interval $[-1,1]$.

Since $(1-t)^d \cdot F_{\triangle}\left(\frac{1+t}{1-t}\right) = h_{\triangle}(t)$,

A new-old real-rootedness result

The F-polynomials of the type B Coxeter complexes have the same coefficients (up to sign) as the *derivative polynomials* $Q_n(x)$ *for secant*, defined by $\frac{d^n}{dx^n}\sec(x) = Q_n(\tan x)\cdot\sec(x)$.

Theorem

All roots of the derivative polynomials for hyperbolic tangent and secant are interlaced, real, and belong to the interval $[-1,1]$.

Since $(1-t)^d \cdot F_\triangle\left(\frac{1+t}{1-t}\right) = h_\triangle(t)$, the h-polynomials of type B Coxeter complexes have only real roots.

A new-old real-rootedness result

The F-polynomials of the type B Coxeter complexes have the same coefficients (up to sign) as the *derivative polynomials* $Q_n(x)$ *for secant*, defined by $\frac{d^n}{dx^n}\sec(x) = Q_n(\tan x)\cdot\sec(x)$.

Theorem

All roots of the derivative polynomials for hyperbolic tangent and secant are interlaced, real, and belong to the interval $[-1,1]$.

Since $(1-t)^d \cdot F_{\triangle}\left(\frac{1+t}{1-t}\right) = h_{\triangle}(t)$, the h-polynomials of type B Coxeter complexes have only real roots. Realized only now, as derivative polynomials for tangent and secant were discussed in connection with *another* Tchebyshev triangulation.

Flag numbers of graded posets

Flag numbers of graded posets

The *upsilon invariant* of a graded poset P of rank $n+1$ is

$$\Upsilon_P(a,b) = \sum_{S \subseteq \{1,\dots,n\}} f_S u_S$$

Here f_S is the number of chains $x_1 < x_2 < \cdots < x_{|S|}$ such that their set of ranks $\{\rho(x_i) : i \in \{1,\dots,|S|\}\}$ is S. The monomial $u_S = u_1 \cdots u_n$ is a monomial in noncommuting variables a and b such that $u_i = b$ for all $i \in S$ and $u_i = a$ for all $i \notin S$.

Flag numbers of graded posets

The *upsilon invariant* of a graded poset P of rank $n+1$ is

$$\Upsilon_P(a, b) = \sum_{S \subseteq \{1,\dots,n\}} f_S u_S$$

Here f_S is the number of chains $x_1 < x_2 < \cdots < x_{|S|}$ such that their set of ranks $\{\rho(x_i) \ : \ i \in \{1, \dots, |S|\}\}$ is S. The monomial $u_S = u_1 \cdots u_n$ is a monomial in noncommuting variables a and b such that $u_i = b$ for all $i \in S$ and $u_i = a$ for all $i \notin S$. The *ab-index* $\Psi_P(a, b) = \sum_S h_S u_S$ defined as $\Upsilon_P(a - b, b)$.

The map $\Psi_P(a, b) \mapsto \Psi_{\widehat{I}(P)}(a, b)$

The map $\Psi_P(a,b) \mapsto \Psi_{\widehat{I(P)}}(a,b)$

It is a linear map.

The map $\Psi_P(a, b) \mapsto \Psi_{\widehat{I(P)}}(a, b)$

It is a linear map. To express it, we need the *Ehrenborg-Readdy coproduct* $\Delta(u) = \sum_{i=1}^{n} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_n$.

The map $\Psi_P(a,b) \mapsto \Psi_{\widehat{I(P)}}(a,b)$

It is a linear map. To express it, we need the *Ehrenborg-Readdy coproduct* $\Delta(u) = \sum_{i=1}^{n} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_n$.

Theorem (Jojić)

$\Psi_{\widehat{I(P)}}(a,b) = \mathcal{I}(\Psi_P(a,b))$, where the linear operator $\mathcal{I} : \mathbb{Q}\langle a,b\rangle \to \mathbb{Q}\langle a,b\rangle$ is defined recursively:

$$\mathcal{I}(u \cdot a) = \mathcal{I}(u) \cdot a + (ab+ba) \cdot u^* + \sum_u \mathcal{I}(u_{(2)}) \cdot ab \cdot u_{(1)}^* \quad (1)$$

$$\mathcal{I}(u \cdot b) = \mathcal{I}(u) \cdot b + (ab+ba) \cdot u^* + \sum_u \mathcal{I}(u_{(2)}) \cdot ba \cdot u_{(1)}^* . \quad (2)$$

The interval transform of the second kind $I_2(P)$

The interval transform of the second kind $I_2(P)$

$I_2(P)$ is the multiset of subposets of $I(P)$ defined as follows: for each $x \in P$ we take the subposets of $I(P)$ formed by all elements $[y, z] \in I(P)$ containing $[x, x]$.

The interval transform of the second kind $I_2(P)$

$I_2(P)$ is the multiset of subposets of $I(P)$ defined as follows: for each $x \in P$ we take the subposets of $I(P)$ formed by all elements $[y, z] \in I(P)$ containing $[x, x]$.

Theorem

$\Psi_{\widehat{I_2}(P)}(a, b) = \mathcal{I}_2(\Psi_P(a, b))$, *where*

$$\mathcal{I}_2(u) = u + u^* + \sum_u M(u^*_{(1)}, u_{(2)}).$$

The interval transform of the second kind $I_2(P)$

$I_2(P)$ is the multiset of subposets of $I(P)$ defined as follows: for each $x \in P$ we take the subposets of $I(P)$ formed by all elements $[y, z] \in I(P)$ containing $[x, x]$.

Theorem

$\Psi_{\widehat{I_2}(P)}(a, b) = \mathcal{I}_2(\Psi_P(a, b))$, *where*

$$\mathcal{I}_2(u) = u + u^* + \sum_u M(u^*_{(1)}, u_{(2)}).$$

Here u^* is the reverse of u and M is the Ehrenborg-Readdy mixing operator satisfying $\Psi_{P\times Q}(a, b) = M(\Psi_P(a, b), \Psi_Q(a, b))$.

The only proof

The only proof

For each $x \in P$, the set of intervals $[y, z]$ contained in $[[x, x], [\widehat{0}, \widehat{1}]] \subset \widehat{I}(P)$ and ordered by inclusion is isomorphic to the direct product $[\widehat{0}, x]^* \times [x, \widehat{1}]$.

Eulerian posets

Eulerian posets

Via Hall's theorem: a graded poset is *Eulerian* if for every interval $[x, y]$ the reduced Euler characteristic of $\triangle((x, y))$ is $(-1)^{\text{rank}([x,y])}$.

Eulerian posets

Via Hall's theorem: a graded poset is *Eulerian* if for every interval $[x, y]$ the reduced Euler characteristic of $\triangle((x, y))$ is $(-1)^{\mathrm{rank}([x,y])}$.

Corollary (Athanasiadis, based on Walker's result)

If P is Eulerian then so is $\widehat{I}(P)$.

Eulerian posets

Via Hall's theorem: a graded poset is *Eulerian* if for every interval $[x, y]$ the reduced Euler characteristic of $\triangle((x, y))$ is $(-1)^{\mathrm{rank}([x,y])}$.

Corollary (Athanasiadis, based on Walker's result)

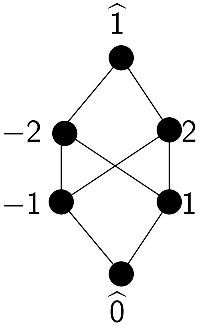
If P is Eulerian then so is $\widehat{I}(P)$.

Theorem (Bayer-Klapper)

For an Eulerian poset P, $\Psi_P(a, b)$ is a polynomial of $c = a + b$ and $d = ab + ba$.

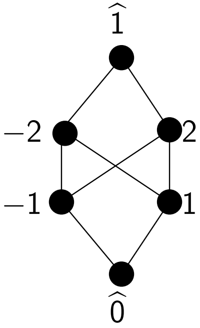
The ladder poset L_n

The ladder poset L_n



L_2

The ladder poset L_n



L_2

$$\Psi_{L_n}(c, d) = c^n.$$

The ladder poset L_n

Theorem (Jojić)

The coefficient of $c^{k_0} d c^{k_1} d \cdots c^{k_r} d c^{k_r}$ *in* $\Psi_{\widehat{I(L_n)}}(c, d)$ *is* $2^r (k_1 + 1)(k_2 + 1) \cdots (k_r + 1)$.

The ladder poset L_n

Theorem (Jojić)

The coefficient of $c^{k_0} d c^{k_1} d \cdots c^{k_r} d c^{k_r}$ in $\Psi_{\widehat{I(L_n)}}(c,d)$ is $2^r (k_1+1)(k_2+1)\cdots(k_r+1)$.

Ehrenborg and Readdy have the dual of this formula for $T(L_n)$.

The ladder poset L_n

Theorem (Jojić)

The coefficient of $c^{k_0} d c^{k_1} d \cdots c^{k_r} d c^{k_r}$ *in* $\Psi_{\widehat{I(L_n)}}(c, d)$ *is* $2^r (k_1 + 1)(k_2 + 1) \cdots (k_r + 1)$.

Ehrenborg and Readdy have the dual of this formula for $T(L_n)$.

Theorem

The coefficient of $c^{k_0} d c^{k_1} d \cdots c^{k_{r-1}} d c^{k_r}$ *in* $\Psi_{I_2(L_n)}(c, d)$ *is* $2^{r+1} (k_0 + 1)(k_1 + 1) \cdots (k_r + 1)$.

The ladder poset L_n

Theorem (Jojić)

The coefficient of $c^{k_0}dc^{k_1}d\cdots c^{k_r}dc^{k_r}$ *in* $\Psi_{\widehat{I}(L_n)}(c,d)$ *is* $2^r(k_1+1)(k_2+1)\cdots(k_r+1)$.

Ehrenborg and Readdy have the dual of this formula for $T(L_n)$.

Theorem

The coefficient of $c^{k_0}dc^{k_1}d\cdots c^{k_{r-1}}dc^{k_r}$ *in* $\Psi_{I_2(L_n)}(c,d)$ *is* $2^{r+1}(k_0+1)(k_1+1)\cdots(k_r+1)$.

The proof involves expressing $M(c^i,c^j)$ as a total weight of lattice paths.

The Boolean algebra $P([1, n])$

The Boolean algebra $P([1, n])$

Lemma

The poset of intervals $\widehat{I}(P([1, n]))$ of the Boolean algebra $P([1, n])$ is isomorphic to the face lattice C_n of the n-dimensional cube.

The Boolean algebra $P([1, n])$

Lemma

The poset of intervals $\widehat{I}(P([1, n]))$ of the Boolean algebra $P([1, n])$ is isomorphic to the face lattice C_n of the n-dimensional cube.

$\Psi_{C_n}(c, d)$ has been expressed by Ehrenborg and Readdy and by Hetyei in terms of (different) signed generalizations of *André-permutations*. Purtill used André permutations, introduced by Foata, Strehl and Schützenberger, to express $\Psi_{P([1,n])}(c, d)$.

The Boolean algebra $P([1, n])$

Lemma

The poset of intervals $\widehat{I}(P([1, n]))$ of the Boolean algebra $P([1, n])$ is isomorphic to the face lattice C_n of the n-dimensional cube.

$\Psi_{C_n}(c, d)$ has been expressed by Ehrenborg and Readdy and by Hetyei in terms of (different) signed generalizations of *André-permutations*. Purtill used André permutations, introduced by Foata, Strehl and Schützenberger, to express $\Psi_{P([1,n])}(c, d)$. $\Psi_{I_2(P([1,n]))}(c, d)$ is an eigenvector of I_2:

$$\Psi_{I_2(P([1,n]))}(c, d) = 2^n \cdot \Psi_{P([1,n])}(c, d).$$

The Boolean algebra $P([1,n])$

Lemma

The poset of intervals $\widehat{I}(P([1,n]))$ of the Boolean algebra $P([1,n])$ is isomorphic to the face lattice C_n of the n-dimensional cube.

$\Psi_{C_n}(c,d)$ has been expressed by Ehrenborg and Readdy and by Hetyei in terms of (different) signed generalizations of *André-permutations*. Purtill used André permutations, introduced by Foata, Strehl and Schützenberger, to express $\Psi_{P([1,n])}(c,d)$. $\Psi_{I_2(P([1,n]))}(c,d)$ is an eigenvector of I_2:

$$\Psi_{I_2(P([1,n]))}(c,d) = 2^n \cdot \Psi_{P([1,n])}(c,d).$$

An analogous result for the Tchebyshev operator of the second kind was obtained by Ehrenborg and Readdy.

Eigenvectors of I_2

Eigenvectors of I_2

Following the blueprint of Ehrenborg and Readdy.

Eigenvectors of I_2

Following the blueprint of Ehrenborg and Readdy.
Lifting: If $u \in \mathbb{Q}\langle a, b\rangle_n$ is an eigenvector of I_2 then so is $\mathcal{L}(u) := (a-b)u + u(a-b) \in \mathbb{Q}\langle a, b\rangle_{n+1}$. Both eigenvectors have the same eigenvalue. (Was $L : u \mapsto (a-b)u$.)

Eigenvectors of I_2

Following the blueprint of Ehrenborg and Readdy.

Lifting: If $u \in \mathbb{Q}\langle a, b\rangle_n$ is an eigenvector of I_2 then so is $\mathcal{L}(u) := (a-b)u + u(a-b) \in \mathbb{Q}\langle a, b\rangle_{n+1}$. Both eigenvectors have the same eigenvalue. (Was $L : u \mapsto (a-b)u$.)

Products: $I_2(P \times Q) = I_2(P)(\times)I_2(Q) \Rightarrow$ if u_1 and u_2 are eigenvectors with eigenvalues λ_1 and λ_2 then so is $M(u_1, u_2)$, with eigenvalue $\lambda_1 \cdot \lambda_2$.

Eigenvectors of I_2

Following the blueprint of Ehrenborg and Readdy.

Lifting: If $u \in \mathbb{Q}\langle a, b\rangle_n$ is an eigenvector of I_2 then so is $\mathcal{L}(u) := (a-b)u + u(a-b) \in \mathbb{Q}\langle a, b\rangle_{n+1}$. Both eigenvectors have the same eigenvalue. (Was $L : u \mapsto (a-b)u$.)

Products: $I_2(P \times Q) = I_2(P)(\times)I_2(Q) \Rightarrow$ if u_1 and u_2 are eigenvectors with eigenvalues λ_1 and λ_2 then so is $M(u_1, u_2)$, with eigenvalue $\lambda_1 \cdot \lambda_2$.

In the case of the Tchebyshev operators of the second kind, all compositions of L and of $u \mapsto M(1, u)$ of length n, applied to 1, yield a basis of eigenvectors for $\mathbb{Q}\langle a, b\rangle_n$.

Eigenvectors of I_2

Following the blueprint of Ehrenborg and Readdy.
Lifting: If $u \in \mathbb{Q}\langle a, b\rangle_n$ is an eigenvector of I_2 then so is $\mathcal{L}(u) := (a-b)u + u(a-b) \in \mathbb{Q}\langle a, b\rangle_{n+1}$. Both eigenvectors have the same eigenvalue. (Was $L : u \mapsto (a-b)u$.)
Products: $I_2(P \times Q) = I_2(P)(\times)I_2(Q) \Rightarrow$ if u_1 and u_2 are eigenvectors with eigenvalues λ_1 and λ_2 then so is $M(u_1, u_2)$, with eigenvalue $\lambda_1 \cdot \lambda_2$.
Now we have a kernel: if $u^* = -u$ then $I_2(u) = 0$.

Eigenvectors of I_2

Following the blueprint of Ehrenborg and Readdy.
Lifting: If $u \in \mathbb{Q}\langle a, b\rangle_n$ is an eigenvector of I_2 then so is $\mathcal{L}(u) := (a-b)u + u(a-b) \in \mathbb{Q}\langle a, b\rangle_{n+1}$. Both eigenvectors have the same eigenvalue. (Was $L : u \mapsto (a-b)u$.)
Products: $I_2(P \times Q) = I_2(P)(\times)I_2(Q) \Rightarrow$ if u_1 and u_2 are eigenvectors with eigenvalues λ_1 and λ_2 then so is $M(u_1, u_2)$, with eigenvalue $\lambda_1 \cdot \lambda_2$.
$\mathbb{Q}\langle a, b\rangle_n = A_{\mathbb{Q}}\langle a, b\rangle_n \oplus S_{\mathbb{Q}}\langle a, b\rangle_n$, where
$A_{\mathbb{Q}}\langle a, b\rangle_n = \{u \in \mathbb{Q}\langle a, b\rangle_n \ : \ u^* = -u\}$ and
$S_{\mathbb{Q}}\langle a, b\rangle_n = \{u \in \mathbb{Q}\langle a, b\rangle_n \ : \ u^* = u\}$.

Eigenvectors of I_2

Following the blueprint of Ehrenborg and Readdy.
Lifting: If $u \in \mathbb{Q}\langle a, b\rangle_n$ is an eigenvector of I_2 then so is $\mathcal{L}(u) := (a-b)u + u(a-b) \in \mathbb{Q}\langle a, b\rangle_{n+1}$. Both eigenvectors have the same eigenvalue. (Was $L : u \mapsto (a-b)u$.)
Products: $I_2(P \times Q) = I_2(P)(\times)I_2(Q) \Rightarrow$ if u_1 and u_2 are eigenvectors with eigenvalues λ_1 and λ_2 then so is $M(u_1, u_2)$, with eigenvalue $\lambda_1 \cdot \lambda_2$.
$\mathbb{Q}\langle a, b\rangle_n = A_{\mathbb{Q}}\langle a, b\rangle_n \oplus S_{\mathbb{Q}}\langle a, b\rangle_n$, where
$A_{\mathbb{Q}}\langle a, b\rangle_n = \{u \in \mathbb{Q}\langle a, b\rangle_n \ : \ u^* = -u\}$ and
$S_{\mathbb{Q}}\langle a, b\rangle_n = \{u \in \mathbb{Q}\langle a, b\rangle_n \ : \ u^* = u\}$.
Conjecture: $A_{\mathbb{Q}}\langle a, b\rangle_n$ is the kernel, and a generating set of eigenvectors for $S_{\mathbb{Q}}\langle a, b\rangle_n$ may be obtained by applying all compositions of length n of $\mathcal{L}$ and of $u \mapsto M(1, u)$ to 1.

THE END

THE END

Thank you very much!

THE END

Thank you very much!
arXiv:2007.07362 [math.CO]

THE END

Thank you very much!
arXiv:2007.07362 [math.CO]