Counting spanning hypertrees and meanders

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What is a hypermap?

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Informally: a hypergraph, topologically embedded in an orientable surface.

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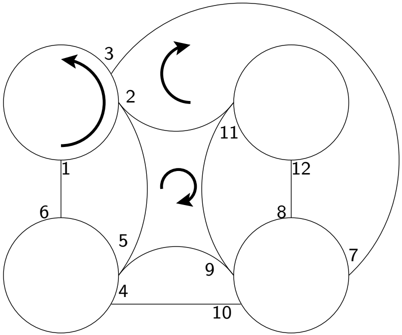

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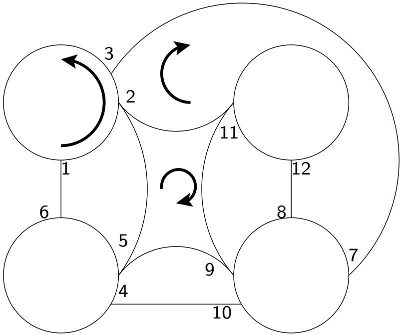

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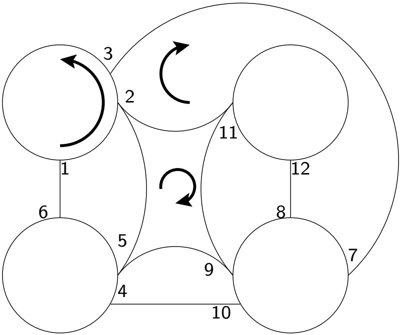

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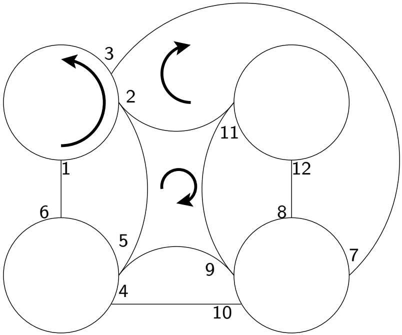

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Faces: $\alpha^{-1}\sigma = (1,5)(2,7,12)(3,6,10)(4,9)(8,11)$.

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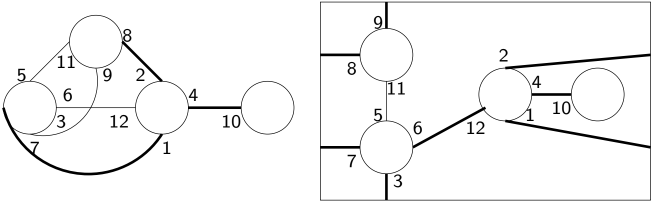

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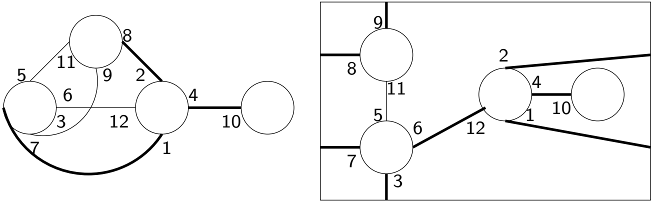

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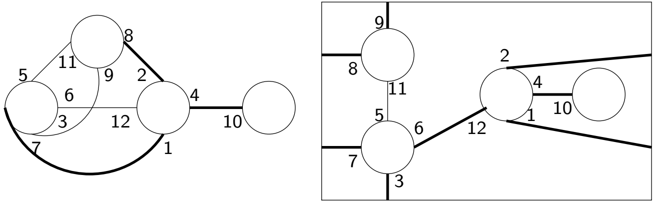

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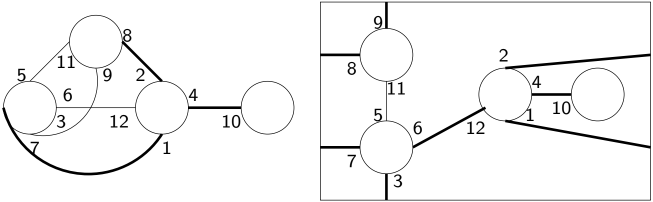

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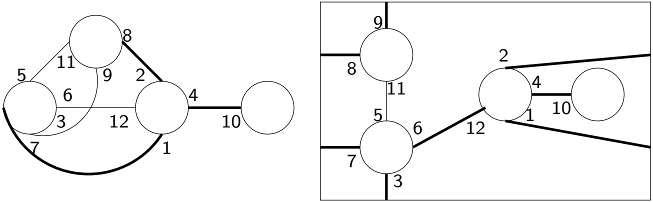

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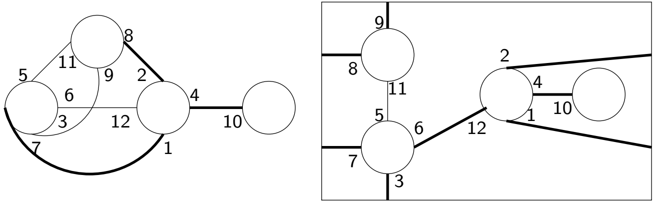

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$12 + 2 - 2 \cdot 1 = 4 + 6 + 2$.

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Write $\alpha = \alpha_1 \cdots \alpha_t$ as a product of disjoint cycles. The permutation $\theta = \theta_1 \cdots \theta_t$ is a *refinement* of α if for each i the permutation θ_i permutes the points moved by α_i and $g(\theta_i, \alpha_i) = 0$. (Equivalently, each θ_i is a noncrossing partition with respect to the cyclic order of α_i.)

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(σ, θ) is a *spanning hypermap* of (σ, α) if θ is a refinement of α.

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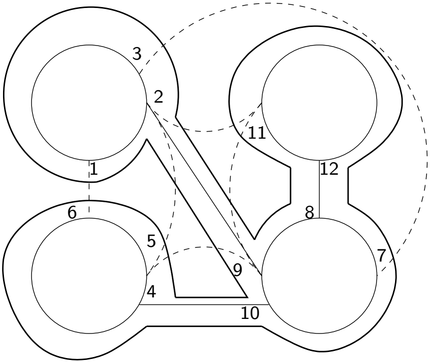

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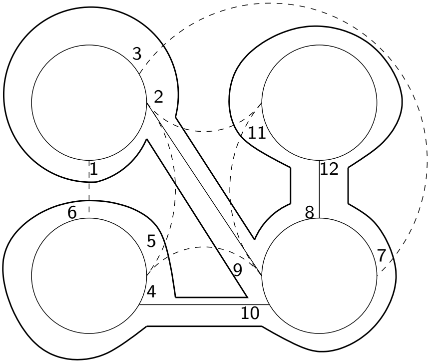

$\theta = (1)(2,9)(3)(4,10)(5)(6)(7)(8,12)(11)$ is a refinement of $\alpha = (1,6)(2,11,9,5)(3,7)(4,10)(8,12)$.

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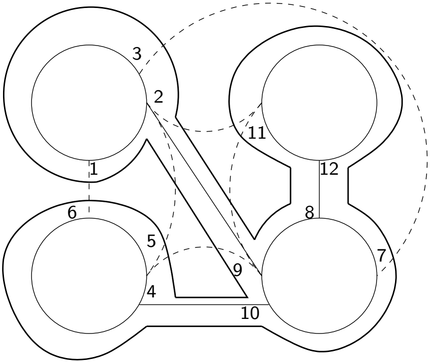

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$\alpha = (1,6)(2,11,9,5)(3,7)(4,10)(8,12)$.
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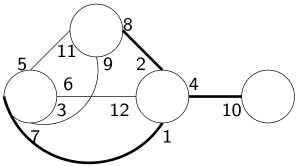

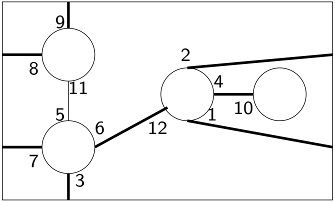

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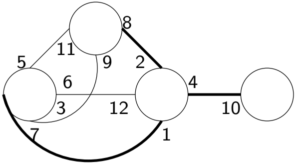

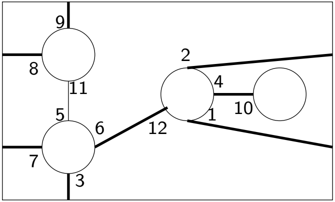

Spanning tree on the left: $\theta_0 = (1,7)(2,8)(4,10)$.

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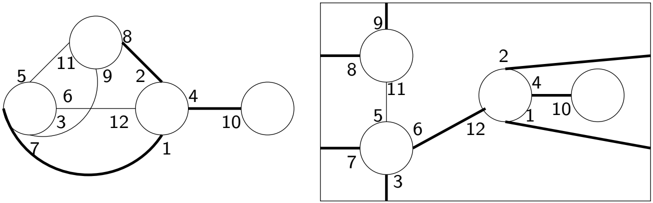

Spanning tree on the left: $\theta_0 = (1,7)(2,8)(4,10)$.
Spanning genus 1 unicellular map on the right:
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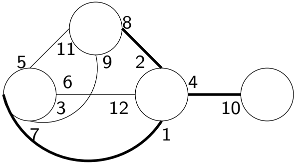

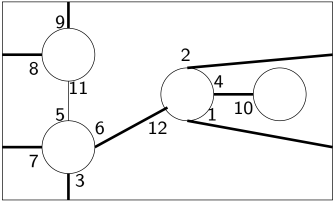

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(We added $(3,9)$ and $(6,12)$.)

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Theorem (Cori ($g = 0$) $\rightarrow$ Machì)

Given (σ, α), there is a bijection between the genus g unicellular hypermaps θ spanning its hyperdual $(\sigma^{-1}, \sigma^{-1}\alpha)$, and the set $C_\sigma(\sigma, \alpha)$, defined as the set of circular permutations ζ satisfying $g(\sigma, \zeta) = g(\sigma, \alpha)$ and $g(\alpha, \zeta) = 0$. The bijection is given by the rule $\theta \mapsto \zeta = \sigma\theta$.

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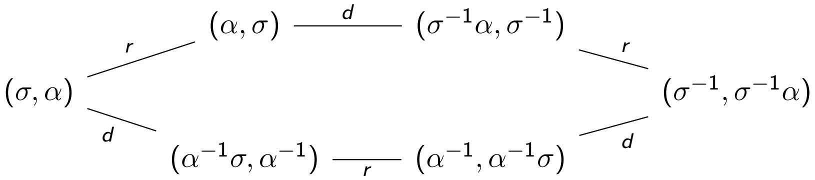

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Corollary

There is a bijection between the spanning genus g unicellular hypermaps θ of a hypermap (σ, α) of genus g and the set

$$C_\sigma(\sigma, \alpha^{-1}\sigma) = \{\zeta : z(\zeta) = 1, g(\sigma, \zeta) = g(\sigma, \alpha^{-1}\sigma), g(\alpha^{-1}\sigma, \zeta) = 0\},$$

taking each spanning unicellular hypermap θ into $\zeta = \theta^{-1}\sigma$.

Machì's theorem and its variants

Bernardi's vertex tour of a spanning tree is also a variant, because of the following.

Theorem

Let (σ, α) be a hypermap and let θ be a permutation of the same set of points. Then (σ, θ) is a spanning unicellular hypermap of (σ, α) if and only if $(\alpha^{-1}\sigma, \alpha^{-1}\theta)$ is a spanning unicellular hypermap of the dual hypermap $(\alpha^{-1}\sigma, \alpha^{-1})$. Furthermore, if the above are satisfied we have

$$g(\sigma, \theta) + g(\alpha^{-1}\sigma, \alpha^{-1}\theta) = g(\sigma, \alpha).$$

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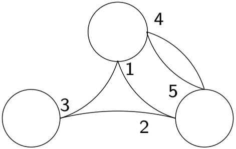
Consider $\sigma = (1,4)(2,5)(3)$ and $\alpha = (1,2,3)(4,5)$.



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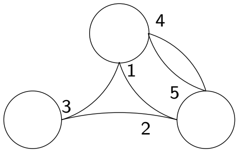
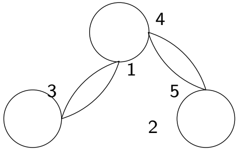


The hyperdeletion (i,j) takes (σ, α) into $(\sigma, \alpha(i,j))$.
For $(i,j) = (1,2)$: $(1,2,3)(1,2) = (1,3)$.



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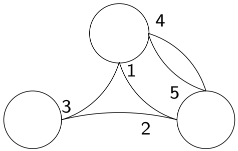
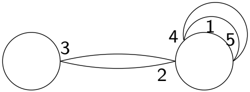


The hypercontraction (i,j) takes (σ, α) into $((i,j)\sigma, (i,j)\alpha)$.
For $(i,j) = (1,2)$: $(1,2)(1,4)(2,5)(3) = (1,4,2,5)(3)$ and $(1,2)(1,2,3) = (2,3)$.



Non-topological hyperdeletions and hypercontractions

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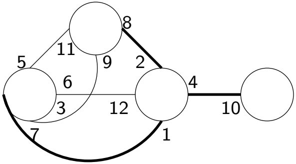

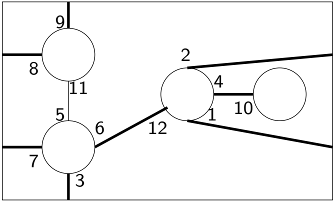

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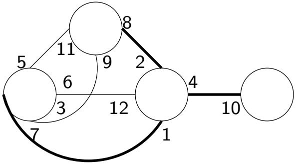

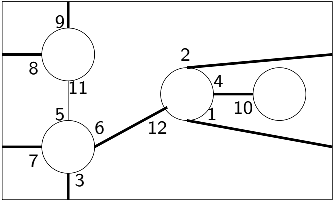

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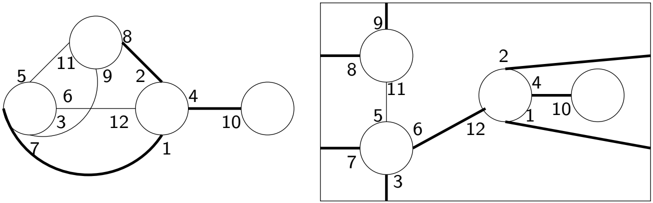

You may delete the "bridge" $(3, 9)$ or $(6, 12)$ from $\theta = (1, 7)(2, 8)(3, 9)(4, 10)(6, 12)$. It will increase the number of faces and $z(\alpha)$ by one, and it will *decrease* the genus by one.

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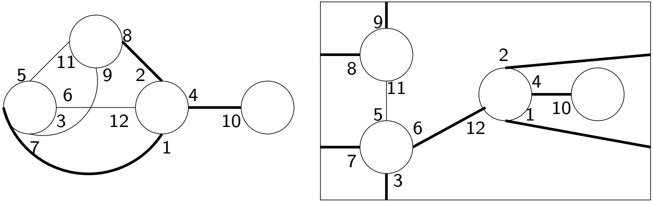

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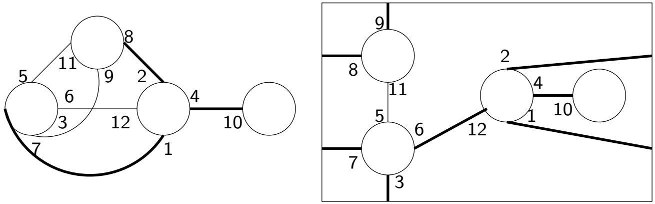

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Two-disk diagrams

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Any (σ, α) may be transformed into a unicellular hypermonopole $(\gamma\sigma, \gamma\alpha\delta) = (\eta, \eta\zeta^{-1})$ of the same genus. Here $\eta = \gamma\alpha$ is the vertex tour and $\zeta = \theta^{-1}\sigma$ (where $\theta = \alpha\delta$) is the face tour.

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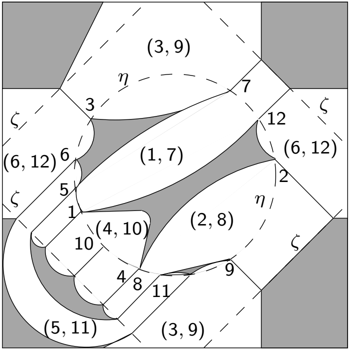

$\sigma = (1, 4, 2, 12)(8, 11, 9)(5, 7, 3, 6)(10)$
$\alpha = (1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12)$
$\gamma = (1, 7)(2, 8)(4, 10)$
$\theta = (1, 7)(2, 8)(3, 9)(4, 10)(6, 12)$
$\eta = (1, 10, 4, 8, 11, 9, 2, 12, 7, 3, 6, 5)$
$\zeta = (1, 10, 4, 8, 11, 3, 12, 7, 9, 2, 6, 5)$
$\eta\zeta^{-1} = (3, 9)(6, 12)$

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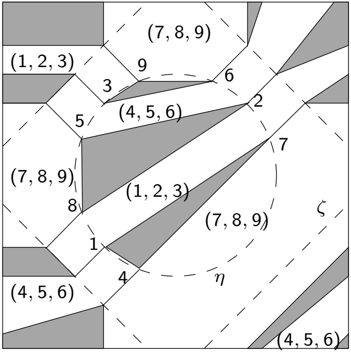

$$\sigma = (1,4,7)(2,5,8)(3,6,9)$$
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$$\theta = (1,2)(5,6)(7,8,9)$$
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$$\eta\zeta^{-1} = (7,8,9)$$

Counting spanning hypertrees

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Theorem

Let $H = (\sigma, \alpha)$ *a hypermap and* $(1, 2, \ldots, m)$ *a cycle of* α*. If* $m \geq 2$ *then the set of all spanning genus g unicellular hypermaps* (σ, θ) *of H is the disjoint union of the following sets* $S_1, S_2, \ldots, S_m$*:*

- S_1 *consists of all spanning genus g unicellular hypermaps of* $H_1 = (\sigma, \alpha(1, m))$*.*

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- *Let $H_2 = ((1,2)\sigma, (1,2)\alpha)$. S_2 consists of all spanning genus g unicellular hypermaps of the form $(\sigma, (1,2)\theta')$, where $((1,2)\sigma, \theta')$ is any spanning genus g (genus $g-1$) unicellular hypermap of H_2 if the hypercontraction of $(1,2)$ is topological (non-topological).*

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- *For $k = 3, \ldots, m$ we set $H_k = ((1, k)\sigma, (1, k)\alpha(1, k-1))$. S_k consists of all genus g unicellular hypermaps $(\sigma, (1, k)\theta')$, where $((1, k)\sigma, \theta')$ is any spanning genus g (genus $g - 1$) unicellular hypermap of the hypermap H_k if the hypercontraction of $(1, k)$ is topological (non-topological).*

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Theorem

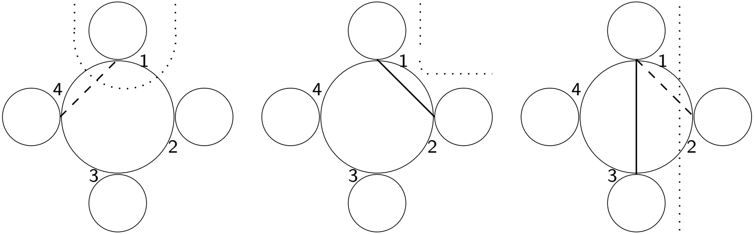
Let $H = (\sigma, \alpha)$ a hypermap and $(1, 2, \ldots, m)$ a cycle of α. If $m \geq 2$ then the set of all spanning genus g unicellular hypermaps (σ, θ) of H is the disjoint union of the following sets $S_1, S_2, \ldots, S_m$:

- *For $k = 3, \ldots, m$ we set $H_k = ((1, k)\sigma, (1, k)\alpha(1, k-1))$. S_k consists of all genus g unicellular hypermaps $(\sigma, (1, k)\theta')$, where $((1, k)\sigma, \theta')$ is any spanning genus g (genus $g-1$) unicellular hypermap of the hypermap H_k if the hypercontraction of $(1, k)$ is topological (non-topological).*

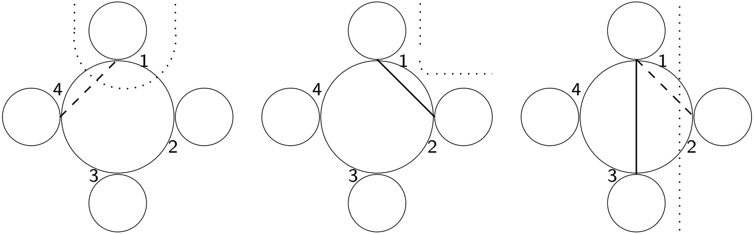
Hint: focus on the second smallest element of the cycle of θ containing 1.

The planar case

The planar case

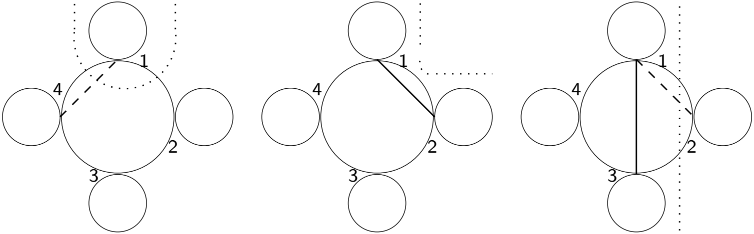


The planar case



$(\sigma, \theta) \in S_k \Leftrightarrow$ the noncrossing partition corresponding to θ belongs to R_k defined by Simion and Ullman as an aid to recursively construct a symmetric chain decomposition of the noncrossing partition lattice.

The planar case



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Semimeanders and reciprocal monopoles

Semimeanders and reciprocal monopoles

Theorem

The number of semimeanders of order n equals the number of spanning hypertrees of the reciprocal of a monopole with $n/2$ nested edges.

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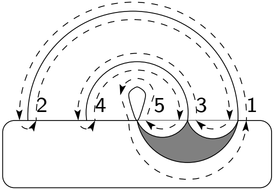

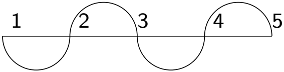

Meanders and reciprocal bipoles

Meanders and reciprocal bipoles

Theorem

The number of meanders of order n equals the number of spanning hypertrees of the reciprocal of a dipole with n parallel edges.

Meanders and reciprocal bipoles

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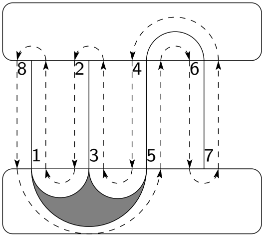

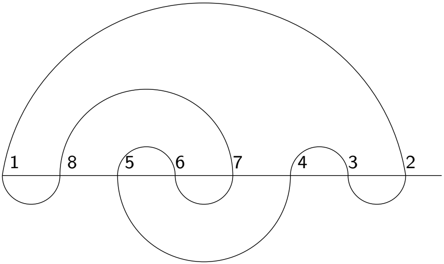

Spanning hypertrees in reciprocals of maps

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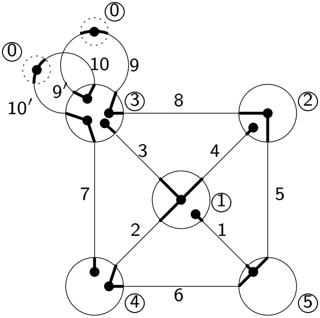
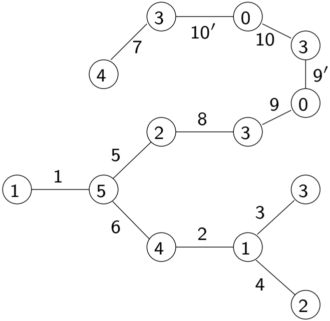
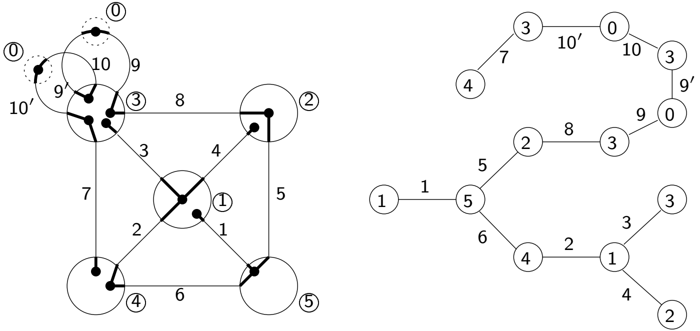


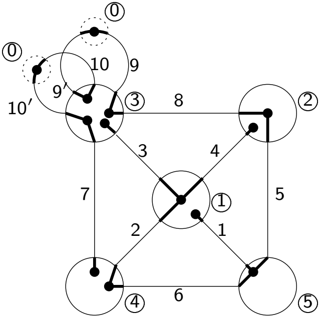


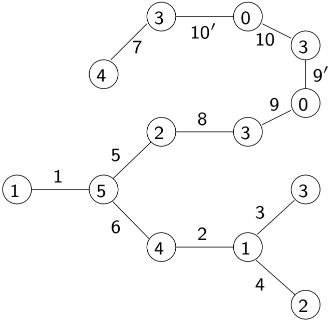
Spanning hypertrees in reciprocals of maps



Generalizing an idea of Franz and Earnshaw (reciprocal analogue of the "tree flipping" $T \mapsto T - \{e\} \cup \{f\}$), it is possible to write and algorithm listing all spanning hypertrees of the reciprocal of a map.

Spanning hypertrees in reciprocals of maps





Generalizing an idea of Franz and Earnshaw (reciprocal analogue of the "tree flipping" $T \mapsto T - \{e\} \cup \{f\}$), it is possible to write and algorithm listing all spanning hypertrees of the reciprocal of a map.

A strange consequence

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For loopless maps with vertices of degree at most three, the number of spanning hypertrees of the reciprocal only depends on the underlying graph and not on the cyclic order of the edges around the vertices.

Thank you!

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arXiv:2110.00176 [math.CO]