Marilyn's Ten-digit Number Problem

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Abstract. In her newspaper column "Ask Marilyn," Marilyn vos Savant was asked why the number $8,549,176,320$ is special (see http://www.marilynvossavant.com/). Her answer was that the number comprises all 10 digits in alphabetical order (eight to zero).

This number also has the property that it is evenly divisible by 2^{10} since its complete factorization is $8,549,176,320 = 2^{10} \cdot 3^3 \cdot 5 \cdot 61,843$. This leads to the following problem.

Find the largest power of 2 that has a ten digit multiple N that contains all of the digits $0, 1, 2, 3, \cdots, 9$ in its decimal representation. Also, for this largest 2^n, find all such N.

Solution: We will solve this problem by using only multiplication and addition on a 10 digit calculator. In order to do this, we will first use Enrico Fermi's method of approximation [1] to very closely estimate what the 2^n that solves the problem has to be. After we do this, we immediately know that we can solve the problem with just a simple calculator. Without using Fermi's method, we would not have the slightest idea what this largest power of 2 would be.

First, suppose a 10 digit positive integer is chosen at random. What is the probability P that this random integer contains all of the 10 digits $0, 1, 2, 3, \cdots, 9$? To solve this, we know that the total number of ten digit positive integers equals $10^{10} - 10^9 = 9 \cdot 10^9$. Also, the total number of ten digit positive integers that contain all of the 10 digits $0, 1, 2, 3, \cdots, 9$ is $9 \cdot 9!$. Thus, the answer is $P = \dfrac{9 \cdot 9!}{9 \cdot 10^9} = \dfrac{9!}{10^9} \approx \dfrac{1}{2756}$.

Let 2^x be a fixed power of 2. A multiple of 2^x is an integer of the form $k \cdot 2^x$ where k is a positive integer. Of course, $k \cdot 2^x$ as k ranges over $k = 1, 2, 3, 4, \cdots$ is the collection of all positive-integer that are divisible by 2^x. Since there are $9 \cdot 10^9$ ten digit integers, we see that the number of multiples of 2^x that are 10 digit integers is roughly equal to $\dfrac{9 \cdot 10^9}{2^x}$.

Using Fermi's method, we will now consider these $\dfrac{9 \cdot 10^9}{2^x}$ multiples of 2^x that give 10 digit integers to be random samples of the 10 digit integers. That is, random in the sense that the 10 digits in each $k \cdot 2^x$ can be considered to be random. Therefore, the integer $k \cdot 2^x$ in each of these random samples has a probability of $\dfrac{1}{2756}$ of containing all of the digits $0, 1, 2, 3, \cdots, 9$. We can now easily estimate what the solution 2^n to the problem is.

Now, $2^{21} = 2,097,152$ and $\dfrac{9 \cdot 10^9}{2^{21}} \approx 4292$.

Also, $2^{22} = 4,194,304$ and $\dfrac{9 \cdot 10^9}{2^{22}} \approx 2146$.

Since $k \cdot 2^{21}, k = 1, 2, 3, \cdots,$ will give us roughly 4292 random samples of the 10 digit integers (i.e., random in the sense that we stated) and since each sample has a probability of $\dfrac{1}{2756}$ of being a 10 digit integer containing all of the digits $0, 1, 2, 3, \cdots, 9$, we can be very confident that some multiple of 2^{21} is a 10 digit integer containing all of the digits $0, 1, 2, 3, \cdots, 9$. Also, for the same reason, we are less confident that 2^{22} will work since 2^{22} only gives 2146 samples.

Also, observe that $0+1+2+3+\cdots+9=45$ and 9 divides 45. Therefore, from elementary number theory, any 10 digit integer that contains all of the digits $0,1,2,3,\cdots,9$ is a multiple of 9. Now, $9\cdot 2^{21}=18,874,368$. Also, $9\cdot 2^{21}\cdot 52=981,467,136$. We now compute $981,467,136+k\,(18,874,368)$ as k ranges over $k=1,2,3,4,\cdots,477$. This is the total range of k that will give 10 digit integers. For each $k=1,2,3,\cdots,477$ we look at the screen of the calculator to see if the 10 digits that are displayed are all different. The standard calculator allows us to do this by just pressing the + key repeatedly. Each time the + key is pressed, the calculator adds 18,874,368. Since the + key must be pressed 477 times, we ourselves solved the problem in about 20 minutes. We found (and double checked it twice) that the only solutions are the following,

(a) $3,076,521,984=9\cdot 2^{21}\cdot 163$.

(b) $3,718,250,496=9\cdot 2^{21}\cdot 197$.

(c) $6,398,410,752=9\cdot 2^{21}\cdot 339$.

Since 163, 197 and 339 are all odd, the answer to the problem is 2^{21} and (a), (b), (c) are the three 10 digit integers that go with 2^{21}.

References

[1] ©1988 by Hans Christian von Baeyer. The Sciences (September/October '88), 2 E. 63 St., New York, N.Y., 10021.

Note Hans Christian von Baeyer is a professor of physics of William and Mary in Williamsburg, VA.