# 2005 MATH Challenge

For full credit you must **justify** your answers.

1. Suppose $f, f'$, and $f''$ are continuous functions on $[0,3]$ satisfying $f(3) = 2$, $f'(3) = 1$ and $\int_0^3 f(x)\,dx = 6$. Find the value of $\int_0^3 x^2 f''(x)\,dx$.

2. Two positive real numbers are given. Their sum is less than their product. Prove that their sum is greater than 4.

3. Let $S$ and $T$ be finite disjoint sets of points of the plane. Prove that there exists a family $L$ of parallel lines such that each point of $S$ belongs to a member of $L$ and no member of $T$ belongs to any member of $L$.

4. A bug starts from the origin on the plane and crawls one unit upwards to $(0,1)$ after one minute. During the second minute, it crawls two units to the right ending at $(2,1)$. Then during the third minute, it crawls three units upward, arriving at $(2,4)$. It makes another right turn and crawls four units during the fourth minute. From here it continues to crawl $n$ units during minute $n$ and then making a $90°$ turn either left or right. The bug continues this until after 16 minutes, it finds itself back at the origin. Its path does not intersect itself. What is the smallest possible area of the 16-gon traced out by its path?

5. Let $f(x) = x^3 + x + 1$ and let $g(x)$ be the inverse function of $f$. Find $g'(3)$.

6. Let $S(n)$ denote the sum of the decimal digits of the integer $n$. For example $S(64) = 10$. Find the smallest integer $n$ such that

$$S(n) + S(S(n)) + S(S(S(n))) = 2007.$$

7. Tom picks a polynomial $p$ with nonnegative integer coefficients. Sally claims that she can ask Tom just two values of $p$ and then tell him all the coefficients. She asks for $p(1)$ and $p(p(1)+1)$. For example, suppose $p(1) = 10$ and $p(11) =$ 46,610. What is the polynomial, and how did she know it?

8. Two integers are called *approximately equal* if their difference is at most 1. How many different ways are there to write 2005 as a sum of one or more positive integers which are all approximately equal to each other? The order of terms does not matter: two ways which only differ in the order of terms are not considered different.

9. An Elongated Pentagonal Orthocupolarotunda is a polyhedron with exactly 37 faces, 15 of which are squares, 7 of which are regular pentagons, and 15 of which are triangles. How many vertices does it have?

10. The bug is back! This time he crawls at a uniform rate, one unit per minute. He starts at the origin at time 0 and crawls one unit to the right, arriving at $(1, 0)$, turns $90°$ left and crawls another unit to $(1, 1)$, turns $90°$ left again, and crawls two units. He continues to make $90°$ left turns as shown in the figure. (The path of the bug establishes a one-to-one correspondence between the non-negative integers and the integer lattice points of the plane.) Let $g(t)$ denote the position in the plane after $t$ minutes, where $t$ is an integer. Thus, for example, $g(0) = (0, 0)$, $g(6) = (-1, -1)$, and $g(16) = (-2, 2)$. Does there exist an integer $t$ such that $g(t)$ and $g(t + 23)$ are exactly 17 units apart? If so, find the smallest such $t$.