2005 MATH Challenge

For full credit you must **justify** your answers.

1. Suppose f, f', and f'' are continuous functions on $[0,3]$ satisfying $f(3) = 2$, $f'(3) = 1$ and $\int_0^3 f(x)\,dx = 6$. Find the value of $\int_0^3 x^2 f''(x)\,dx$.

   Solution: Use integration by parts twice to get

$$\begin{aligned}
\int_0^3 x^2 f''(x)\,dx &= x^2 f'(x)|_0^3 - \int_0^3 f'(x)\cdot 2x\,dx \\
&= x^2 f'(x)|_0^3 - \left[f(x)\cdot 2x - \int_0^3 2f(x)\,dx\right] \\
&= x^2 f'(x)|_0^3 - f(x)\cdot 2x|_0^3 + 2\int_0^3 f(x)\,dx \\
&= 9f'(3) - 6f(3) + 2\cdot 6 \\
&= 9\cdot 1 - 6\cdot 2 + 2\cdot 6 = 9
\end{aligned}$$

2. Two positive real numbers are given. Their sum is less than their product. Prove that their sum is greater than 4.

   Solution: Let the two numbers be x and y. Then $x + y < xy$. By the Arithmetic Mean-Geometric Mean Inequality, $xy < \left(\frac{x+y}{2}\right)^2$, which implies that $x + y < (x + y)^2/4$, from which the result follows.

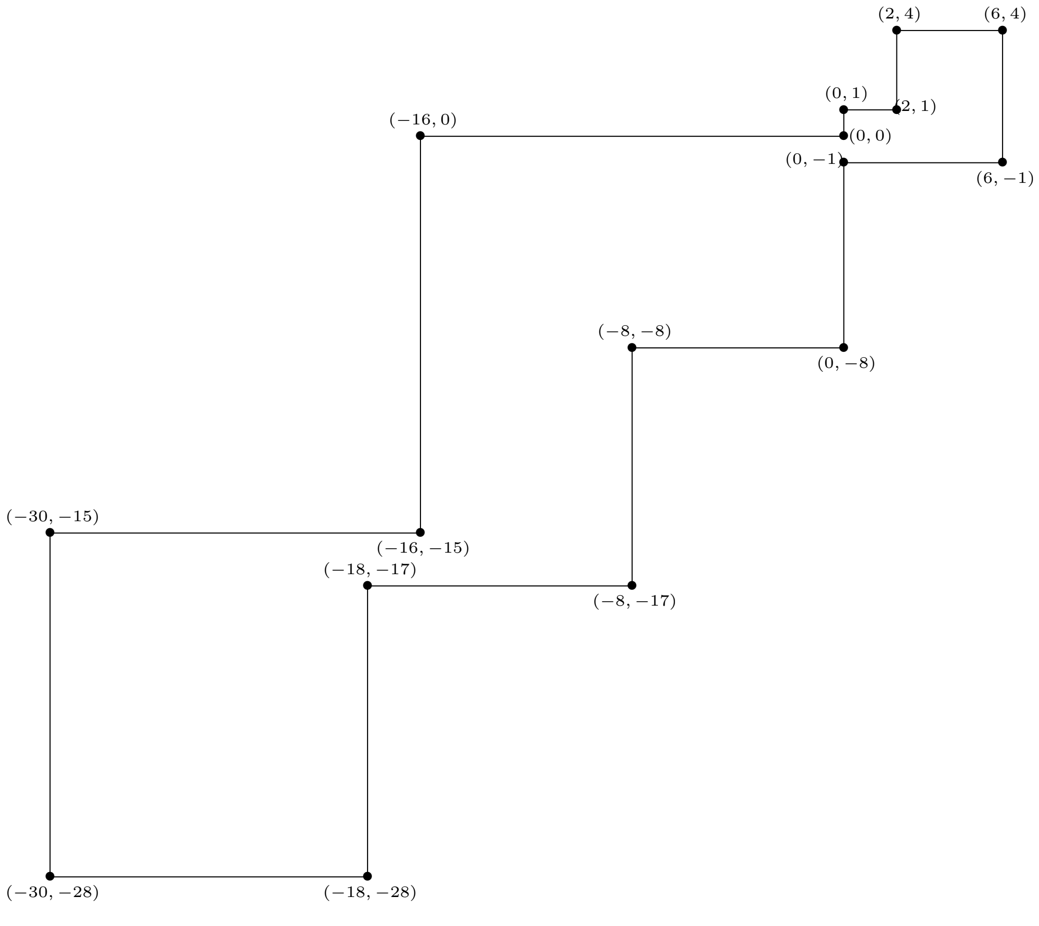
   Or, $(x - y)^2 \geq 0$ so $x^2 + 2xy + y^2 \geq 4xy$. Thus $(x + y)/2 \geq \sqrt{xy}$. Combine this with $x+y < xy$ to get (letting $u = x+y$), $u < xy < (u/2)^2$ which implies $0 < 4u < u^2$ and the inequality follows from this.

3. Let S and T be finite disjoint sets of points of the plane. Prove that there exists a family L of parallel lines such that each point of S belongs to a member of L and no member of T belongs to any member of L.

   Solution: Solution. Let R denote the collection of slopes of lines joining the pairs $(P, Q), P \in S, Q \in T$. (This may include ∞.) Next pick any real number r not in R. The family of lines with slope r through points of S satisfies the required conditions.

4. A bug starts from the origin on the plane and crawls one unit upwards to $(0, 1)$ after one minute. During the second minute, it crawls two units to the right ending at $(2, 1)$. Then during the third minute, it crawls three units upward, arriving at $(2, 4)$. It makes another right turn and crawls four units during the fourth minute. From here it continues to crawl n units during minute n and then making a $90°$ turn either left or right. The bug continues this until after 16 minutes, it finds itself back at the origin. Its path does not intersect itself. What is the smallest possible area of the 16-gon traced out by its path?

Solution: 384. Note that the vertical sides of the polygon are all odd lengths. Since the up sides equals the down sides, we must partition the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ into two subsets, one containing both 1 and 3 so that the sum of the members of each subset is $(1+3+5+7+9+11+13+15) \div 2 = 32$. Experimentation shows that this can be done in only one way: $Up = \{1, 3, 13, 15\}$ and $Down = \{5, 7, 9, 11\}$. The horizontal edges are trickier. The sum $2 + 4 + 6 + \cdots + 16 = 72$ so the partition must be into two sets each with sum 36 such that the 'right' edges include both 2 and 4. This can be done in four ways: $\{2, 4, 14, 16|6, 8, 10, 12\}, \{2, 4, 8, 10, 12|6, 14, 16\}, \{2, 4, 6, 8, 16|10, 12, 14\}$, $\{2, 4, 6, 10, 14|8, 12, 16\}$. Only two of these give rise to non-intersecting paths. The path for $Right = \{2, 4, 14, 16\}, Left = \{6, 8, 10, 12\}$ is shown below. The other nonintersecting path is given by the partition $Right = \{2, 4, 6, 8, 16\}$, $Left = \{10, 12, 14\}$, $Up = \{1, 3, 13, 15\}$ and $Down = \{5, 7, 9, 11\}$, which gives rise to a 16-gon whose area is 660.


(2, 4)
(6, 4)
(0, 1)
(2, 1)
(−16, 0)
(0, 0)
(0, −1)
(6, −1)
(−8, −8)
(0, −8)
(−30, −15)
(−16, −15)
(−18, −17)
(−8, −17)
(−30, −28)
(−18, −28)

5. Let $f(x) = x^3 + x + 1$ and let $g(x)$ be the inverse function of f. Find $g'(3)$.

Solution: The function g satisfies $f \circ g(x) = x$ for all x, so $\frac{d}{dx}\ f \circ g(x) = f'(g(x)) \cdot g'(x) = \frac{d}{dx}\ x = 1$ be the chain rule. Thus $g'(3) = 1/f'(g(3))$. Since $f'(x) = 3x^2 + 1$ it follows that $f'(1) = 4$. Also, $g(3) = f^{-1}(3) = 1$ since $f(1) = 3$. Finally $g'(3) = 1/f'(1) = 1/4$.

6. Let $S(n)$ denote the sum of the decimal digits of the integer n. For example $S(64) = 10$. Find the smallest integer n such that

$$S(n) + S(S(n)) + S(S(S(n))) = 2007.$$

Solution: Since $S(n) \leq 9 \log n$, it follows that n must have at least 200 digits. In fact if n has few than 220 digits, then $S(n) \leq 219 \cdot 9 = 1971$ and $S(S(n)) \leq 1 + 9 + 6 + 9 = 25$ and $s(S(S(n))) \leq 10$, in which case their sum is at most 2006. Since $n \equiv S(n)(\text{mod } 9)$, and $2007 \equiv 0(\text{mod } 9)$, it follows that any solution n satisfies one of $n \equiv 0(\text{mod } 9)$, $n \equiv 3(\text{mod } 9)$, or $n \equiv 6(\text{mod } 9)$. We are left to try multiples of 3 whose sum of digits is at least 1971. Trying 1971, 1974, 1977, 1980, 1983, etc, we see that 1977 is the smallest integer that works. The other $S(n)$ values that work are 1980, 1983, and 2001. The smallest number for which $S(n) = 1977$ is a digit 6 followed by a sting of two-hundred and nineteen 9's. That is $n = 6 \cdot 10^{219} + 10^{219} - 1 = 7 \cdot 10^{219} - 1$.

7. Tom picks a polynomial p with nonnegative integer coefficients. Sally claims that she can ask Tom just two values of p and then tell him all the coefficients. She asks for $p(1)$ and $p(p(1)+1)$. For example, suppose $p(1) = 10$ and $p(11) = 46{,}610$. What is the polynomial, and how did she know it?

Solution: His first answer 10 is the sum of the coefficients of p. She knows that $p(1)$ is at least as big as any of p's coefficients. If $p(x) = ax^4 + bx^3 + cx^2 + dx + e$, then $p(11) = a11^4 + b11^3 + c11^2 + d11 + e$ which is just the base 11 representation of $p(11)$. Expressing 46,610 in base 11 notation yields 32023_{11}, so $p(x) = 3x^4 + 2x^3 + 2x + 3$.

8. Two integers are called *approximately equal* if their difference is at most 1. How many different ways are there to write 2005 as a sum of one or more positive integers which are all approximately equal to each other? The order of terms does not matter: two ways which only differ in the order of terms are not considered different.

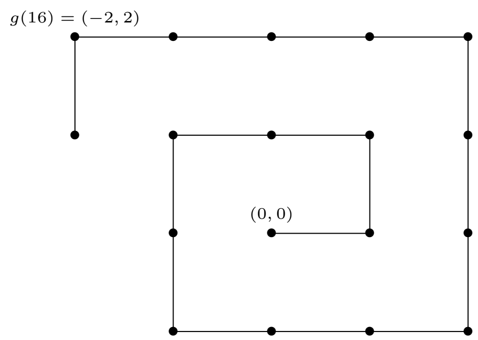
   Solution: 2005. Let $G(n)$ denote the number of ways to write n as a sum of approximately equal positive integers. Trial and error produces $G(1) = 1, G(2) = 2, G(3) = 3$ and $G(4) = 4$. In fact, we can prove by mathematical induction that $G(n) = n$ for all positive integers. Suppose $n = a_1 + a_2 + \cdots a_k$ where either all the a_i are the same or there are two different values and they differ by 1. Thus we have either $n = ka + m(a-1), m > 0$ or $n = ka$. In the first case $n+1 = (k+1)a + (m-1)(a-1)$ if $m > 0$ and $n+1 = a+1+(k-1)a$. So we have a bijection between the representations of n and the representations of $n+1$ that include at least one number other that 1. In addition $n+1 = 1+1+1+\cdots+1$ represents a new representation. Thus $G(n+1) = G(n)+1$ for all integers n.

   Alternatively, note that for each n, $1 \leq n \leq 2005$ there is exactly one sum with n terms. To see this, divide 2005 by n. If $2005 = nq + r$, then $2005 = (n-r)q + r(q+1)$. Of course q and $q+1$ are approximately equal.

9. An Elongated Pentagonal Orthocupolarotunda is a polyhedron with exactly 37 faces, 15 of which are squares, 7 of which are regular pentagons, and 15 of which are triangles. How many vertices does it have?

   Solution: The number of edges is $\frac{1}{2}(15 \cdot 4 + 7 \cdot 5 + 15 \cdot 3) = 70$, so by Euler's formula $e + 2 = f + v$, we have $70 + 2 = 37 + v$ and $v = 35$.

10. The bug is back! This time he crawls at a uniform rate, one unit per minute. He starts at the origin at time 0 and crawls one unit to the right, arriving at $(1,0)$, turns $90°$ left and crawls another unit to $(1,1)$, turns $90°$ left again, and crawls two units. He continues to make $90°$ left turns as shown in the figure. (The path of the bug establishes a one-to-one correspondence between the non-negative integers and the integer lattice points of the plane.) Let $g(t)$ denote the position in the plane after t minutes, where t is an integer. Thus, for example, $g(0) = (0,0)$, $g(6) = (-1,-1)$, and $g(16) = (-2,2)$. Does there exist an integer t such that $g(t)$ and $g(t+23)$ are exactly 17 units apart? If so, find the smallest such t.



Solution: The path from $g(t)$ to $g(t+23)$ must involve exactly one turn. The only solutions to $u^2 + (23-u)^2 = 17^2$ are $u = 8$ and $u = 15$. The first straight edge of length 15 occurs on the left to right segment from $g(210) = (-7,-7)$ to $g(225) = (8,-7)$. Going back 8 units to $g(202) = (-7,1)$, we see that

$$D((-7,1),(8,-7)) = \sqrt{15^2 + 8^2} = 17.$$

Incidentally, the function g is given by

$$g(t) = \begin{cases} (t - 4n^2 - 3n, -n) & \text{if } 2n(2n+1) \leq t < (2n+1)^2 \\ (n+1, t - 4n^2 - 5n - 1) & \text{if } (2n+1)^2 \leq t < (2n+1)(2n+2) \\ (4n^2 - n - t, n) & \text{if } 0 < (2n-1)(2n+1) \leq t < (2n)^2 \\ (-n, 4n^2 + n - t) & \text{if } 0 < (2n)^2 \leq t < (2n)(2n+1) \end{cases}$$