Eight Versions of n-Pile Nim

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Abstract. We first state eight versions of n-pile Bouton's nim. Then we briefly review the concept of balanced and unbalanced positions in a game. After introducing some simple definitions, we determine balanced and unbalanced positions for all eight versions, including the regular n-pile Bouton's nim. We define the concept of dual games. We then point out that these eight versions occur in duals and that the balanced and unbalanced positions in dual games are also duals. We leave it to the reader to supply all of the reasonably easy proofs since most of the readers will enjoy doing this. We also believe many readers will be surprised at how mind-twisting some of these simple concepts can be.

Eight versions of Bouton's nim. A move in n-pile Bouton's nim consists of two steps, (a) the moving player selects a non-empty pile, and (b) the moving player decides how many counters to remove from it. If one or both of these options is given to the opposing player, the nature of the game changes greatly. In this paper, we consider all possible variations of this type together with the misère versions of these games. All eight games start with $n \geq 1$ non-empty piles of counters. Two players, Art and Beth, alternate moving. All eight games are symmetric or impartial. This means that if the rules for Art's move have been specified, then the rules for Beth's move are defined by interchanging the words Art and Beth in the rules for Art's move. The eight games we consider are the following:
$\begin{pmatrix} 12 & \overline{1}2 & 1\overline{2} & \overline{1}\,\overline{2} \\ \overline{\overline{12}} & \overline{\overline{\overline{1}2}} & \overline{\overline{1\overline{2}}} & \overline{\overline{\overline{1}\,\overline{2}}} \end{pmatrix}$ where 1 means means that the first player chooses a pile and $\overline{1}$ means the pile is chosen for him by his opponent. The second symbol refers to the way the number of counters is chosen, 2 means the moving player chooses, $\overline{2}$ means the number in chosen by the non-moving player. The double bar is the misère version. For example, $\overline{\overline{1\overline{2}}}$ is the game in which Art chooses his own pile, then Beth takes some counters for him, etc. and the loser is the last player to make a legal move.

Game 12 **(Bouton's nim)** Art's move is defined as follows.

1. Art chooses any non-empty pile.

2. Art removes any number of counters from the chosen pile.

By symmetry Beth's move is defined by interchanging the words Art and Beth in Art's move. The winner is the last player who moves; that is, the player who removes the last counter.

Game $1\overline{2}$ (move reverse nim) Art's move is defined as follows.

1. Art chooses any non-empty pile.
2. Beth removes any number of counters from the chosen pile.

The winner is the player whose move results in the last counter being removed. Thus the player who physically removes the last counter is the loser.

Game $\overline{1}2$ (pile reverse nim) Art's move is defined as follows.

1. Beth chooses any non-empty pile.
2. Art removes any number of counters that he wishes from the chosen pile.

The winner is the player whose move results in the last counter being removed. Thus, the player who physically removes the last counter is the winner.

Game $\overline{1}\ \overline{2}$ (totally reverse nim) Art's move is defined as follows.

1. Beth chooses any non-empty pile.
2. Beth removes any number of counters from the chosen pile.

The winner is the player whose move results in the last counter being removed. Of course, the player who physically removes the last counter is the loser.

Game $\overline{12}$ (misère game 12**)** The rules for game $\overline{12}$ are the same as game 12 except the loser is the player who removes the last counter. This game is the misère Bouton's nim.

Definition. Two games G and G' are called duals provided that conditions (1) and (2) are satisfied:

1. In the two definitions for Art's move, the words Art and Beth have been interchanged. This means that when Art moves in game G', he is moving by the same rules that Beth uses for moving in game G and vice-versa.
2. The definition of the winner-loser has been reversed. This means that in one game the winner is the player whose move results in the last counter being removed while in the other game the loser is the player whose move results in the last counter being removed.

Thus, the following pairs of games are duals: $\overline{12}$ and $\overline{1}\ \overline{2}$; $1\overline{2}$ and $\overline{\overline{1}2}$; $\overline{1}2$ and $\overline{1\overline{2}}$; and $\overline{1}\ \overline{2}$ and $\overline{12}$.

Game $\overline{1\overline{2}}$ (misère game $1\overline{2}$) The rules for game $\overline{1\overline{2}}$ are the same as game $1\overline{2}$ except the loser is the player whose move results in the last counter being removed.

Game $\overline{\overline{1}2}$ (misère game $\overline{1}2$) The rules for game $\overline{\overline{1}2}$ are the same as game $\overline{1}2$ except the loser is the player whose move results in the last counter being removed.

Game $\overline{\overline{1}\ \overline{2}}$ (misère game $\overline{1}\ \overline{2}$) The rules for game $\overline{\overline{1}\ \overline{2}}$ are the same as game $\overline{1}\ \overline{2}$ except the loser is the player who removes the last counter.

We will now introduce some notation.

Notation Suppose there are k piles having $n_1, n_2, \cdots, n_k$ counters. Then we can denote this position by $p = (n_1, n_2, \cdots, n_k)$ where we do not distinguish between the different permutations of $n_1, n_2, \cdots, n_k$.

1. The nim value of a position p is defined as $n(p) = n(n_1, n_2, \cdots, n_k) = n_1 \oplus n_2 \oplus \cdots \oplus n_k$ where $\oplus$ is the nim sum. The nim sum $n_i \oplus n_j$ is obtained by writing the integers n_i, n_j in binary and adding modulo 2 without carrying. For example,

$$26 \oplus 12 = 11010_2 \oplus 1100_2 = 10110_2 = 22.$$

   For the position $p = (n_1, n_2, \cdots, n_k)$,

2. $g(p)$ denotes the number of piles that have two or more counters (called *great* piles).

3. $u(p)$ denotes the number of singleton (or *unit*) piles.

It is well-known that the set N of non-negative integers is an abelian group under $\oplus$ satisfying $a \oplus b \leq a + b$.

Balanced and unbalanced positions Suppose G denotes any one of our eight games. For each G we will specify the balanced and unbalanced positions of that game. These balanced and unbalanced positions must have the following properties, where $0 = (0, 0, \cdots, 0)$ denotes the terminal position.

1. Suppose $p = (n_1, n_2, \cdots, n_k)$ is any non-terminal balanced position. By symmetry suppose Art's move starts at p. Then Beth by using perfect play can force Art's move to end in an unbalanced position.

2. Suppose p is any non-terminal unbalanced position, and suppose Art's move starts at p. Then Art by using perfect play can force his move to end in a balanced position.

3. If G is a regular game (i.e., winner makes the last move), then 0 is balanced.

4. If G is a misère game (i.e., loser makes the last move), then 0 is unbalanced.

We will denote the set of balanced, unbalanced positions by $\mathcal{B}$,$\mathcal{U}$ respectively.

Figure 1 goes here.

Theorem 1 *Suppose $\mathcal{B}$,$\mathcal{U}$ have been determined for a game G, and let p_0 denote the initial position of the game. Then the first moving player can win with perfect play if $p_0 \in \mathcal{U}$, and the second moving player can win with perfect play if $p_0 \in \mathcal{B}$.*

Proof. The player who is destined to win merely forces the positions in the game to alternate between $\mathcal{B}$,$\mathcal{U}$ until he wins. ■

Theorem 2 *Suppose G and G' are dual games. Then $\mathcal{B}_G = \mathcal{U}_{G'}$ and $\mathcal{U}_G = \mathcal{B}_{G'}$.*

We leave the proof to the reader. Note that in some ways the theorem is self-evident.

In theorems 3-6 and their corollaries, we specify $\mathcal{B}$,$\mathcal{U}$ for each of our eight versions of nim. The proofs are fairly easy, and we let the reader prove all of them. The corollary of each theorem follows by applying theorem 2. The reader should use Fig 1 when supplying the proofs.

Theorem 3. *The balanced and unbalanced positions of game 12 are as follows,*

1. *A position $p = (n_1, n_2, \cdots, n_k)$ is balanced if $n(p) = n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$.*
2. *A position p is unbalanced if $n(p) \neq 0$.*

Game 12 is the standard Bouton's nim.

Corollary. The balanced and unbalanced positions of game $\overline{\overline{1}\,\overline{2}}$ are as follows,

1. A position p is balanced if $n(p) \neq 0$.
2. A position p is unbalanced if $n(p) = 0$.

Theorem 4. *The balanced and unbalanced positions of game $\overline{12}$ are as follows,*

1. *A position p is balanced if either*

   a' *$g(p) \geq 1$ and $n(p) = 0$ or*

   b' *$g(p) = 0$ and $n(p) = 1$.*

2. *A position p is unbalanced if either*

(a) $g(p) \geq 1$ *and* $n(p) \neq 0$ *or*

(b) $g(p) = 0$ *and* $n(p) = 0$.

Corollary. The balanced and unbalanced positions in Game D are as follows,

1. A position p is balanced if either

   (a) $g(p) \geq 1$ and $n(p) \neq 0$ or

   (b) $g(p) = 0$ and $n(p) = 0$.

2. A position p is unbalanced if either

   (a) $g(p) \geq 1$ and $n(p) = 0$ or

   (b) $g(p) = 0$ and $n(p) = 1$.

Theorem 5. *In the game $1\overline{2}$, a position p is balanced if $u(p)$ is even, and p is unbalanced if $u(p)$ is odd.*

Corollary. In the game $\overline{\overline{1}2}$, a position p is balanced if $u(p)$ is odd and p is unbalanced if $u(p)$ is even.

Theorem 6. *In the game $\overline{1}2$, a position p is balanced if either $g(p) \geq 1$ and $u(p)$ is odd or $g(p) = 0$ and $u(p)$ is even. Also p is unbalanced if either $g(p) \geq 1$ and $u(p)$ is even or $g(p) = 0$ and $u(p)$ is odd.*

Corollary. In the game $\overline{\overline{1}2}$, a position p is balanced if either $g(p) \geq 1$ and $u(p)$ is even or $g(p) = 0$ and $u(p)$ is odd. A position p is unbalanced if either $g(p) \geq 1$ and $u(p)$ is odd or $g(p) = 0$ and $u(p)$ is even.

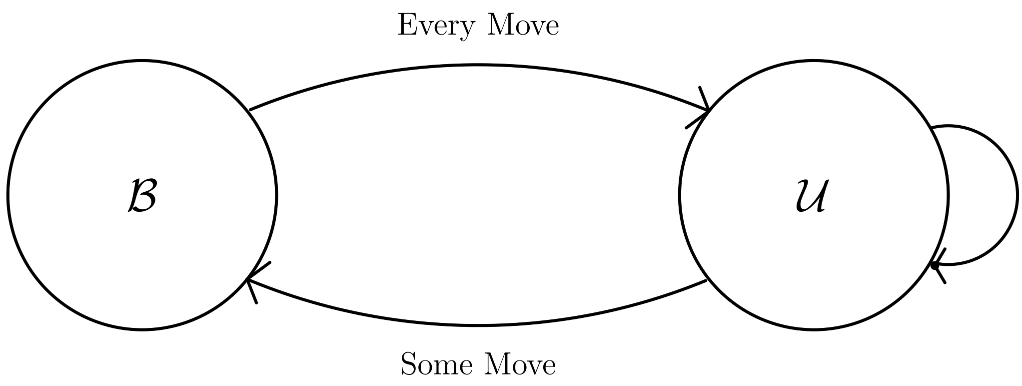

Fig. 1

Notice that in the regular game $0 \in \mathcal{B}$, but in the misère version, $0 \in \mathcal{U}$.

References

[1] Berlekamp, Conway, and Guy, Winning Ways, Academic Press, New York, 1982.

[2] Richard K. Guy, Fair Game, 2nd ed., COMAP, New York, 1989.