1. **(E)** Only the rectangle that goes in position II must match on both vertical sides. Since rectangle D is the only one for which these matches exist, it must be the one that goes in position II. Hence the rectangle that goes in position I must be E.

2 9 E 7 0	5 7 D 4 8	1 4 A 6 9
0 1 B 3 6	8 3 C 5 2	

2. **(E)** We need to make the numerator large while making the denominator small. The smallest the denominator can be is $0 + 1 = 1$. The largest the numerator can be is $9+8 = 17$. The fraction $17/1$ is an integer, so $A+B = 17$.

3. **(D)** The subtraction problem posed is equivalent to the addition problem

$$\begin{array}{r} \texttt{4 8 b} \\ \underline{+\ \texttt{c 7 3}} \\ \texttt{7 a 2} \end{array}$$

which is easier to solve. Since $\texttt{b} + 3 = 12$, $\texttt{b}$ must be 9. Since $1 + 8 + 7$ has units digit $\texttt{a}$, $\texttt{a}$ must be 6. Because $1 + 4 + \texttt{c} = 7$, $\texttt{c} = 2$. Hence $\texttt{a} + \texttt{b} + \texttt{c} = 6 + 9 + 2 = 17$.

4. **(E)** Notice that the operation has the property that, for any r, a, b, and c,

$$[ra, rb, rc] = \frac{ra + rb}{rc} = [a, b, c].$$

Thus all three of the expressions $[60, 30, 90], [2, 1, 3],$ and $[10, 5, 15]$ have the same value, which is 1. So $\big[[60, 30, 90], [2, 1, 3], [10, 5, 15]\big] = [1, 1, 1] = 2$.

5. **(C)** Factor the left side of the given equation:

$$2^{1998} - 2^{1997} - 2^{1996} + 2^{1995} = (2^3 - 2^2 - 2 + 1)2^{1995} = 3 \cdot 2^{1995} = k \cdot 2^{1995},$$

so $k = 3$.

6. **(C)** The number 1998 has prime factorization $2 \cdot 3^3 \cdot 37$. It has eight factor-pairs: $1 \times 1998 = 2 \times 999 = 3 \times 666 = 6 \times 333 = 9 \times 222 = 18 \times 111 = 27 \times 74 = 37 \times 54 = 1998$. Among these, the smallest difference is $54 - 37 = 17$.

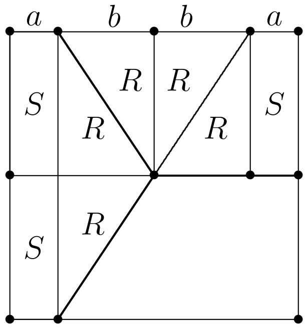
7. **(D)** $\sqrt[3]{N\sqrt[3]{N\sqrt[3]{N}}} = \sqrt[3]{N\sqrt[3]{N \cdot N^{\frac{1}{3}}}} = \sqrt[3]{N\sqrt[3]{N^{\frac{4}{3}}}} = \sqrt[3]{N \cdot N^{\frac{4}{9}}} = \sqrt[3]{N^{\frac{13}{9}}} = N^{\frac{13}{27}}$.

OR

$$\sqrt[3]{N\sqrt[3]{N\sqrt[3]{N}}} = \left(N\left(N(N)^{\frac{1}{3}}\right)^{\frac{1}{3}}\right)^{\frac{1}{3}} = \left(N\left(N^{\frac{1}{3}} \cdot N^{\frac{1}{9}}\right)\right)^{\frac{1}{3}} = N^{\frac{1}{3}} \cdot N^{\frac{1}{9}} \cdot N^{\frac{1}{27}} = N^{\frac{13}{27}}.$$

8. **(D)** The area of each trapezoid is 1/3, so $\frac{1}{2} \cdot \frac{1}{2}(x + \frac{1}{2}) = \frac{1}{3}$. Simplifying yields $x + \frac{1}{2} = \frac{4}{3}$, and it follows that $x = 5/6$.

OR



$2S + 2R = S + 3R$
$\therefore S = R$
$\therefore b = 2a$
$a + b + b + a = 1$
$\therefore 2b + a = 5/6$

9. **(D)** Let N be the number of people in the audience. Then $0.2N$ people heard 60 minutes, $0.1N$ heard 0 minutes, $0.35N$ heard 20 minutes, and $0.35N$ heard 40 minutes. In total, the N people heard

$$60(0.2N) + 0(0.1N) + 20(0.35N) + 40(0.35N) = 12N + 0 + 7N + 14N = 33N$$

minutes, so they heard an average of 33 minutes each.

10. **(A)** Let x and y denote the dimensions of the four congruent rectangles. Then $2x + 2y = 14$, so $x + y = 7$. The area of the large square is $(x + y)^2 = 7^2 = 49$.

11. **(D)** The four vertices determine six possible diameters, namely, the four sides and two diagonals. However, the two diagonals are diameters of the same circle. Thus there are five circles.

12. **(A)** Note that $N = 7^{5^{3^{2^{11}}}}$, which has only 7 as a prime factor.

13. **(E)** Factor 144 into primes, $144 = 2^4 \cdot 3^2$, and notice that there are at most two 6's and no 5's among the numbers rolled. If there are no 6's, then there must be two 3's since these are the only values that can contribute 3 to the prime factorization. In this case the four 2's in the factorization must be the result of two 4's in the roll. Hence the sum $3 + 3 + 4 + 4 = 14$ is a possible value for the sum. Next consider the case with just one 6. Then there must be one 3, and the three remaining 2's must be the result of a 4 and a 2. Thus, the sum $6 + 3 + 4 + 2 = 15$ is also possible. Finally, if there are two 6's, then there must also be two 2's or a 4 and a 1, with sums of $6+6+2+2 = 16$ and $6 + 6 + 4 + 1 = 17$. Hence 18 is the only sum not possible.

OR

Since 5 does not divide 144 and $6^3 > 144$, there can be no 5's and at most two 6's. Thus the only ways the four dice can have a sum of 18 are: $4, 4, 4, 6$; $2, 4, 6, 6$; and $3, 3, 6, 6$. Since none of these products is 144, the answer is (E).

14. **(A)** Because the parabola has x-intercepts of opposite sign and the y-coordinate of the vertex is negative, a must be positive, and c, which is the y-intercept, must be negative. The vertex has x-coordinate $-b/2a = 4 > 0$, so b must be negative.

15. **(C)** The regular hexagon can be partitioned into six equilateral triangles, each with area one-sixth of the original triangle. Since the original equilateral triangle is similar to each of these, and the ratio of the areas is 6, it follows that the ratio of the sides is $\sqrt{6}$.

16. **(B)** The area of the shaded region is

$$\frac{\pi}{2}\left(\left(\frac{a+b}{2}\right)^2 + \left(\frac{a}{2}\right)^2 - \left(\frac{b}{2}\right)^2\right) = \frac{\pi}{2}\frac{a+b}{2}\left(\frac{a+b}{2} + \frac{a-b}{2}\right) = \frac{\pi(a+b)a}{4}$$

and the area of the unshaded region is

$$\frac{\pi}{2}\left(\left(\frac{a+b}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2\right) = \frac{\pi}{2}\frac{a+b}{2}\left(\frac{a+b}{2} + \frac{b-a}{2}\right) = \frac{\pi(a+b)b}{4}.$$

Their ratio is a/b.

17. **(E)** Note that $f(x) = f(x + 0) = x + f(0) = x + 2$ for any real number x. Hence $f(1998) = 2000$. The function defined by $f(x) = x + 2$ has both properties: $f(0) = 2$ and $f(x + y) = x + y + 2 = x + (y + 2) = x + f(y)$.

OR

Note that

$$2 = f(0) = f(-1998 + 1998) = -1998 + f(1998).$$

Hence $f(1998) = 2000$.

18. **(A)** Suppose the sphere has radius r. We can write the volumes of the three solids as functions of r as follows:

$$\text{Volume of cone} = A = \frac{1}{3}\pi r^2(2r) = \frac{2}{3}\pi r^3,$$

$$\text{Volume of cylinder} = M = \pi r^2(2r) = 2\pi r^3, \text{ and}$$

$$\text{Volume of sphere} = C = \frac{4}{3}\pi r^3.$$

Thus, $A - M + C = 0$.

Note. The AMC logo is designed to show this classical result of Archimedes.

19. **(C)** The area of the triangle is $\frac{1}{2}$(base)(height) $= \frac{1}{2} \cdot (5 - (-5)) \cdot |5 \sin\theta| = 25|\sin\theta|$. There are four values of θ between 0 and 2π for which $|\sin\theta| = 0.4$, and each value corresponds to a distinct triangle with area 10.

OR

The vertex $(5\cos\theta, 5\sin\theta)$ lies on a circle of diameter 10 centered at the origin. In order that the triangle have area 10, the altitude from that vertex must be 2. There are four points on the circle that are 2 units from the x-axis.

20. **(C)** There are eight ordered triples of numbers satisfying the conditions: $(1, 2, 10), (1, 3, 9), (1, 4, 8), (1, 5, 7), (2, 3, 8), (2, 4, 7), (2, 5, 6)$, and $(3, 4, 6)$. Because Casey's card gives Casey insufficient information, Casey must have seen a 1 or a 2. Next, Tracy must not have seen a $6, 9$, or 10, since each of these would enable Tracy to determine the other two cards. Finally, if Stacy had seen a 3 or a 5 on the middle card, Stacy would have been able to determine the other two cards. The only number left is 4, which leaves open the two possible triples $(1, 4, 8)$ and $(2, 4, 7)$.

21. **(C)** Let r be Sunny's rate. Thus $\dfrac{h}{r}$ and $\dfrac{h+d}{r}$ are the times it takes Sunny to cover h meters and $h+d$ meters, respectively. Because Windy covers only $h-d$ meters while Sunny is covering h meters, it follows that Windy's rate is $\dfrac{(h-d)r}{h}$. While Sunny runs $h+d$ meters, the number of meters Windy runs is $\dfrac{(h-d)r}{h} \cdot \dfrac{h+d}{r} = h - \dfrac{d^2}{h}$. Sunny's victory margin over Windy is $\dfrac{d^2}{h}$.

22. **(C)** Express each term using a base-10 logarithm, and note that the sum equals

$$\log 2/\log 100! + \log 3/\log 100! + \cdots + \log 100/\log 100! = \log 100!/\log 100! = 1.$$

OR

Since $1/\log_k 100!$ equals $\log_{100!} k$ for all positive integers k, the expression equals $\log_{100!}(2 \cdot 3 \cdot \cdots \cdot 100) = \log_{100!} 100! = 1$.

23. **(D)** Complete the squares in the two equations to bring them to the form

$$(x-6)^2 + (y-3)^2 = 7^2 \qquad \text{and} \qquad (x-2)^2 + (y-6)^2 = k + 40.$$

The graphs of these equations are circles. The first circle has radius 7, and the distance between the centers of the circles is 5. In order for the circles to have a point in common, therefore, the radius of the second circle must be at least 2 and at most 12. It follows that $2^2 \le k + 40 \le 12^2$, or $-36 \le k \le 104$. Thus $b - a = 140$.

24. **(C)** There are 10,000 ways to write the last four digits $\mathtt{d}_4\mathtt{d}_5\mathtt{d}_6\mathtt{d}_7$, and among these there are $10000 - 10 = 9990$ for which not all the digits are the same. For each of these, there are exactly two ways to adjoin the three digits $\mathtt{d}_1\mathtt{d}_2\mathtt{d}_3$ to obtain a memorable number. There are ten memorable numbers for which the last four digits are the same, for a total of $2 \cdot 9990 + 10 = 19990$.

OR

Let A denote the set of telephone numbers for which $\mathtt{d}_1\mathtt{d}_2\mathtt{d}_3$ and $\mathtt{d}_4\mathtt{d}_5\mathtt{d}_6$ are identical and B the set for which $\mathtt{d}_1\mathtt{d}_2\mathtt{d}_3$ is the same as $\mathtt{d}_5\mathtt{d}_6\mathtt{d}_7$. A number $\mathtt{d}_1\mathtt{d}_2\mathtt{d}_3$-$\mathtt{d}_4\mathtt{d}_5\mathtt{d}_6\mathtt{d}_7$ belongs to $A \cap B$ if and only if $\mathtt{d}_1 = \mathtt{d}_4 = \mathtt{d}_5 = \mathtt{d}_2 = \mathtt{d}_6 = \mathtt{d}_3 = \mathtt{d}_7$. Hence, $n(A \cap B) = 10$. Thus, by the *Inclusion-Exclusion Principle*,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 10^3 \cdot 1 \cdot 10 + 10^3 \cdot 10 \cdot 1 - 10 = 19990.$$

25. **(B)** The crease in the paper is the perpendicular bisector of the segment that joins (0,2) to (4,0). Thus the crease contains the midpoint $(2,1)$ and has slope 2, so the equation $y = 2x-3$ describes it. The segment joining (7,3) and (m,n) must have slope $-\frac{1}{2}$, and its midpoint $\left(\frac{7+m}{2}, \frac{3+n}{2}\right)$ must also satisfy the equation $y = 2x-3$. It follows that

$$-\frac{1}{2} = \frac{n-3}{m-7} \qquad \text{and} \qquad \frac{3+n}{2} = 2 \cdot \frac{7+m}{2} - 3, \text{ so}$$

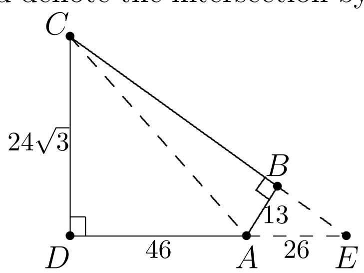
$$2n + m = 13 \qquad \text{and} \qquad n - 2m = 5.$$

Solve these equations simultaneously to find that $m = 3/5$ and $n = 31/5$, so that $m + n = 34/5 = 6.8$.

OR

As shown above, the crease is described by the equation $y = 2x-3$. Therefore, the slope of the line through (m,n) and $(7,3)$ is $-1/2$, so the points on the line can be described parametrically by $(x,y) = (7-2t, 3+t)$. The intersection of this line with the crease $y = 2x-3$ is found by solving $3+t = 2(7-2t)-3$. This yields the parameter value $t = 8/5$. Since $t = 8/5$ determines the point on the crease, use $t = 2(8/5)$ to find the coordinates $m = 7-2(16/5) = 3/5$ and $n = 3 + (16/5) = 31/5$.

26. **(B)** Extend $\overline{DA}$ through A and $\overline{CB}$ through B and denote the intersection by E. Triangle ABE is a 30°-60°-90° triangle with $AB = 13$, so $AE = 26$. Triangle CDE is also a 30°-60°-90° triangle, from which it follows that $CD = (46+26)/\sqrt{3} = 24\sqrt{3}$. Now apply the *Pythagorean Theorem* to triangle CDA to find that $AC = \sqrt{46^2 + (24\sqrt{3})^2} = 62$.



OR

Since the opposite angles sum to a straight angle, the quadrilateral is cyclic, and AC is the diameter of the circumscribed circle. Thus AC is the diameter of the circumcircle of triangle ABD. By the *Extended Law of Sines*,

$$AC = \frac{BD}{\sin 120°} = \frac{BD}{\sqrt{3}/2}.$$

We determine BD by the *Law of Cosines*:

$$BD^2 = 13^2 + 46^2 + 2 \cdot 13 \cdot 46 \cdot \frac{1}{2} = 2883 = 3 \cdot 31^2, \text{ so } BD = 31\sqrt{3}.$$

Hence $AC = 62$.

27. **(E)** After step one, twenty $3 \times 3 \times 3$ cubes remain, eight of which are corner cubes and twelve of which are edge cubes. At this stage each $3 \times 3 \times 3$ corner cube contributes 27 units of area and each $3 \times 3 \times 3$ edge cube contributes 36 units of area. The second stage of the tunneling process takes away 3 units of area from each of the eight $3 \times 3 \times 3$ corner cubes (1 for each exposed surface), but adds 24 units to the area (4 units for each of the six 1×1 center facial cubes removed). The twelve $3 \times 3 \times 3$ edge cubes each lose 4 units but gain 24 units. Therefore, the total surface area of the figure is

$$8 \cdot (27 - 3 + 24) + 12 \cdot (36 - 4 + 24) = 384 + 672 = 1056.$$

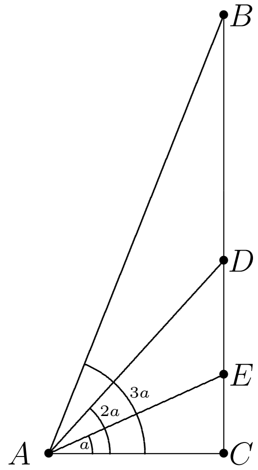
28. **(B)** Let E denote the point on $\overline{BC}$ for which $\overline{AE}$ bisects $\angle CAD$. Because the answer is not changed by a similarity transformation, we may assume that $AC = 2\sqrt{5}$ and $AD = 3\sqrt{5}$. Apply the *Pythagorean Theorem* to triangle ACD to obtain $CD = 5$, then apply the *Angle Bisector Theorem* to triangle CAD to obtain $CE = 2$ and $ED = 3$. Let $x = DB$. Apply the Pythagorean Theorem to triangle ACE to obtain $AE = \sqrt{24}$, then apply the Angle Bisector Theorem to triangle EAB to obtain $AB = (x/3)\sqrt{24}$. Now apply the Pythagorean Theorem to triangle ABC to get

$$(2\sqrt{5})^2 + (x + 5)^2 = \left(\frac{x}{3}\sqrt{24}\right)^2,$$

from which it follows that $x = 9$. Hence $BD/DC = 9/5$, and $m + n = 14$.

OR

Denote by a the measure of angle CAE. Let $AC = 2u$, and $AD = 3u$. It follows that $CD = \sqrt{5}u$. We may assume $BD = \sqrt{5}$. (Otherwise, we could simply modify the triangle with a similarity transformation.) Hence, the ratio CD/BD we seek is just u. Since $\cos 2a = 2/3$, we have $\sin a = 1/\sqrt{6}$. Applying the *Law of Sines* in triangle ABD yields



$$\frac{\sin D}{AB} = \frac{\sin a}{\sqrt{5}} = \frac{2/3}{\sqrt{(2u)^2 + (\sqrt{5}(1+u))^2}} = \frac{1/\sqrt{6}}{\sqrt{5}}.$$

Solve this for u to get

$$\begin{aligned} 2\sqrt{5}\sqrt{6} &= 3\sqrt{4u^2 + 5(1 + 2u + u^2)} \\ 120 &= 9(9u^2 + 10u + 5) \\ 0 &= 27u^2 + 30u - 25 \\ 0 &= (9u - 5)(3u + 5) \end{aligned}$$

so $u = 5/9$ and $m + n = 14$.

OR

Again, let $a = \angle CAE$. We are given that $\cos 2a = 2/3$ and we wish to compute

$$\frac{CD}{BD} = \frac{AC \tan 2a}{AC(\tan 3a - \tan 2a)} = \left(\frac{\tan 3a}{\tan 2a} - 1\right)^{-1}.$$

Let $y = \tan a$. Trigonometric identities yield (upon simplification)

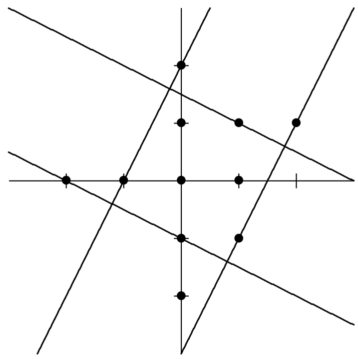
$$\left(\frac{\tan 3a}{\tan 2a} - 1\right)^{-1} = \frac{2(1 - 3y^2)}{(1 + y^2)^2} \qquad \text{and} \qquad \frac{2}{3} = \cos 2a = \frac{1 - y^2}{1 + y^2}.$$

Thus $y^2 = 1/5$ and

$$\frac{CD}{BD} = \frac{2(1 - 3/5)}{(6/5)^2} = \frac{5}{9}.$$

Alternatively, starting with $a = \cos^{-1}(2/3)/2$, electronic calculation yields $\tan(3a)/\tan(2a) = 2.8 = 14/5$, so $CD/BD = 5/9$.

29. **(D)** If a square encloses three collinear lattice points, then it is not hard to see that the square must also enclose at least one additional lattice point. It therefore suffices to consider squares that enclose only the lattice points (0,0), (0,1), and (1,0). If a square had two adjacent sides, neither of which contained a lattice point, then the square could be enlarged slightly by moving those sides parallel to themselves. To be largest, therefore, a square must contain a lattice point on at least two non-adjacent sides. The desired square will thus have parallel sides that contain (1,1) and at least one of $(-1,0)$ and $(0,-1)$. The size of the square is determined by the separation between two parallel sides. Because the distance between parallel lines through (1,1) and $(0,-1)$ can be no larger than $\sqrt{5}$, the largest conceivable area for the square is 5. To see that this is in fact possible, draw the lines of slope 2 through $(-1,0)$ and $(1,-1)$, and the lines of slope $-1/2$ through $(1,1)$ and $(0,-1)$. These four lines can be described by the equations $y = 2x + 2$, $y = 2x - 3$, $2y + x = 3$, and $2y + x = -2$, respectively. They intersect to form a square whose area is **5**, and whose vertices are $(-1/5, 8/5)$, $(9/5, 3/5)$, $(4/5, -7/5)$, and $(-6/5, -2/5)$. There are only three lattice points inside this square.



30. **(E)** Factor a_n as a product of prime powers:

$$a_n = n(n+1)(n+2)\cdots(n+9) = 2^{e_1}3^{e_2}5^{e_3}\cdots.$$

Among the ten factors $n, n+1, \ldots, n+9$, five are even and their product can be written $2^5m(m+1)(m+2)(m+3)(m+4)$. If m is even then $m(m+2)(m+4)$ is divisible by 16 and thus $e_1 \geq 9$. If m is odd, then $e_1 \geq 8$. If $e_1 > e_3$, then the rightmost nonzero digit of a_n is even. If $e_1 \leq e_3$, then the rightmost nonzero digit of a_n is odd. Hence we seek the smallest n for which $e_3 \geq e_1$. Among the ten numbers $n, n+1, \ldots, n+9$, two are divisible by 5 and at most one of these is divisible by 25. Hence $e_3 \geq 8$ if and only if one of $n, n+1, \ldots, n+9$ is divisible by 5^7. The smallest n for which a_n satisfies $e_3 \geq 8$ is thus $n = 5^7 - 9$, but in this case the product of the five even numbers among $n, n+1, \ldots, n+9$ is $2^5m(m+1)(m+2)(m+3)(m+4)$ where m is even, namely $(5^7-9)/2 = 39058$. As noted earlier, this gives $e_1 \geq 9$. For $n = 5^7 - 8 = 78117$, the product of the five even numbers among $n, n+1, \ldots, n+9$ is $2^5m(m+1)(m+2)(m+3)(m+4)$ with $m = 39059$. Note that in this case $e_1 = 8$. Indeed, $39059+1$ is divisible by 4 but not by 8, and $39059+3$ is divisible by 2 but not by 4. Compute the rightmost nonzero digit as follows. The odd numbers among $n, n+1, \ldots, n+9$ are $7811\underline{7}, 7811\underline{9}, 7812\underline{1}, 7812\underline{3}, 78125 = 5^7$ and the product of the even numbers $78118, 78120, 78122, 78124, 78126$ is $2^5 \cdot 39059 \cdot 39060 \cdot 39061 \cdot 39062 \cdot 39063 = 2^5 \cdot 3905\underline{9} \cdot (2^2 \cdot 5 \cdot 195\underline{3}) \cdot 3906\underline{1} \cdot (2 \cdot 1953\underline{1}) \cdot 3906\underline{3}$. (For convenience, we have underlined the needed unit digits.) Having written $n(n+1)\cdots(n+9)$ as 2^85^8 times a product of odd factors not divisible by 5, we determine the rightmost nonzero digit by multiplying the units digits of these factors. It follows that, for $n = 5^7 - 8$, the rightmost nonzero digit of a_n is the units digit of $7 \cdot 9 \cdot 1 \cdot 3 \cdot 9 \cdot 3 \cdot 1 \cdot 1 \cdot 3 = (9 \cdot 9) \cdot (7 \cdot 3) \cdot (3 \cdot 3)$, namely 9.