1. **(E)** Pairing the first two terms, the next two terms, etc. yields

$$\begin{aligned}&1-2+3-4+\cdots-98+99=\\&(1-2)+(3-4)+\cdots+(97-98)+99=\\&-1-1-1-\cdots-1+99=50,\end{aligned}$$

since there are 49 of the -1's.

OR

$$\begin{aligned}&1-2+3-4+\cdots-98+99=\\&1+[(-2+3)+(-4+5)+\cdots+(-98+99)]=\\&1+[1+1+\cdots+1]=1+49=50.\end{aligned}$$

2. **(A)** Triangles with side lengths of $1,1,1$ and $2,2,2$ are equilateral and not congruent, so (A) is false. Statement (B) is true since all triangles are convex. Statements (C) and (E) are true since each interior angle of an equilateral triangle measures $60°$. Furthermore, all three sides of an equilateral triangle have the same length, so (D) is also true.

3. **(E)** The desired number is the arithmetic average or mean:

$$\frac{1}{2}\left(\frac{1}{8}+\frac{1}{10}\right)=\frac{1}{2}\cdot\frac{18}{80}=\frac{9}{80}.$$

4. **(A)** A number one less than a multiple of 5 is has a units digit of 4 or 9. A number whose units digit is 4 cannot be one greater than a multiple of 4. Thus, it is sufficient to examine the numbers of the form $10d+9$ where d is one of the ten digits. Of these, only $9, 29, 49, 69$ and 89 are one greater than a multiple of 4. Among these, only 29 and 89 are prime and their sum is 118.

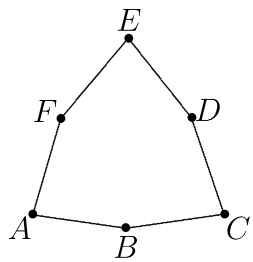
5. **(C)** If the suggested retail price was P, then the marked price was $0.7P$. Half of this is $0.35P$, so Alice paid 35% of the suggested retail price.

6. **(D)** Note that

$$2^{1999}\cdot 5^{2001}=2^{1999}\cdot 5^{1999}\cdot 5^2=10^{1999}\cdot 25=25\overbrace{0\ldots0}^{1999\text{ zeros}}.$$

Hence the sum of the digits is 7.

7. **(B)** The sum of the angles in a convex hexagon is 720° and each angle must be less than 180°. If four of the angles are acute, then their sum would be less than 360°, and therefore at least one of the two remaining angles would be greater than 180°, a contradiction. Thus there can be at most three acute angles. The hexagon shown has three acute angles, A, C, and E.



OR

The result holds for *any* convex n-gon. The sum of the exterior angles of a convex n-gon is 360°. Hence at most three of these angles can be obtuse, for otherwise the sum would exceed 360°. Thus the largest number of acute angles in any convex n-gon is three.

8. **(D)** Let w and $2w$ denote the ages of Walter and his grandmother, respectively, at the end of 1994. Then their respective years of birth are $1994 - w$ and $1994 - 2w$. Hence $(1994 - w) + (1994 - 2w) = 3838$, and it follows that $w = 50$ and Walter's age at the end of 1999 will be 55.

9. **(D)** The next palindromes after 29792 are $29892, 29992, 30003$, and 30103. The difference $30103 - 29792 = 311$ is too far to drive in three hours without exceeding the speed limit of 75 miles per hour. Ashley could have driven $30003 - 29792 = 211$ miles during the three hours for an average speed of $70\frac{1}{3}$ miles per hour.

10. **(C)** Since both I and III cannot be false, the digit must be 1 or 3. So either I or III is the false statement. Thus II and IV must be true and (C) is necessarily correct. For the same reason, (E) must be incorrect. If the digit is 1, (B) and (D) are incorrect, and if the digit is 3, (A) is incorrect.

11. **(A)** The locker labeling requires $137.94/0.02 = 6897$ digits. Lockers 1 through 9 require 9 digits, lockers 10 through 99 require $2 \cdot 90 = 180$ digits, and lockers 100 through 999 require $3 \cdot 900 = 2700$ digits. Hence the remaining lockers require $6897 - 2700 - 180 - 9 = 4008$ digits, so there must be $4008/4 = 1002$ more lockers, each using four digits. In all, there are $1002 + 999 = 2001$ student lockers.

12. **(C)** The x-coordinates of the intersection points are precisely the zeros of the polynomial $p(x) - q(x)$. This polynomial has degree at most three, so it has at most three zeros. Hence, the graphs of the fourth degree polynomial functions intersect at most three times. Finding an example to show that three intersection points can be achieved is left to the reader.

13. **(C)** Since $a_{n+1} = \sqrt[3]{99} \cdot a_n$ for all $n \geq 1$, it follows that $a_1, a_2, a_3, \ldots$ is a geometric sequence whose first term is 1 and whose common ratio is $r = \sqrt[3]{99}$. Thus

$$a_{100} = a_1 \cdot r^{100-1} = \left(\sqrt[3]{99}\right)^{99} = 99^{33}.$$

14. **(A)** Tina and Alina each sang either 5 or 6 times. If N denotes the number of songs sung by trios, then $3N = 4+5+5+7 = 21$ or $3N = 4+5+6+7 = 22$ or $3N = 4+6+6+7 = 23$. Since the girls sang as trios, the total must be a multiple of 3. Only 21 qualifies. Therefore, $N = 21/3 = 7$ is the number of songs the trios sang.

    Challenge. Devise a schedule for the four girls so that each one sings the required number of songs.

15. **(E)** From the identity $1 + \tan^2 x = \sec^2 x$ it follows that $1 = \sec^2 x - \tan^2 x = (\sec x - \tan x)(\sec x + \tan x) = 2(\sec x + \tan x)$, so $\sec x + \tan x = 0.5$.

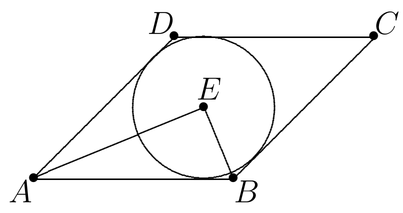
    OR

    The given relation can be written as $\dfrac{1-\sin x}{\cos x} = 2$. Squaring both sides yields $\dfrac{(1-\sin x)^2}{1-\sin^2 x} = 4$, hence $\dfrac{1-\sin x}{1+\sin x} = 4$. It follows that $\sin x = -\dfrac{3}{5}$ and that

$$\cos x = \frac{1-\sin x}{2} = \frac{1-(-3/5)}{2} = \frac{4}{5}.$$

    Thus $\sec x + \tan x = \frac{5}{4} - \frac{3}{4} = 0.5$.

16. **(C)** Let E be the intersection of the diagonals of a rhombus $ABCD$ satisfying the conditions of the problem. Because these diagonals are perpendicular and bisect each other, $\triangle ABE$ is a right triangle with sides 5, 12, and 13 and area 30. Therefore the altitude drawn to side AB is $60/13$, which is the radius of the inscribed circle centered at E.



17. **(C)** From the hypothesis, $P(19) = 99$ and $P(99) = 19$. Let

$$P(x) = (x - 19)(x - 99)Q(x) + ax + b,$$

where a and b are constants and $Q(x)$ is a polynomial. Then

$$99 = P(19) = 19a + b \text{ and } 19 = P(99) = 99a + b.$$

It follows that $99a - 19a = 19 - 99$, hence $a = -1$ and $b = 99 + 19 = 118$. Thus the remainder is $-x + 118$.

18. **(E)** Note that the range of $\log x$ on the interval $(0, 1)$ is the set of all negative numbers, infinitely many of which are zeros of the cosine function. In fact, since $\cos(x) = 0$ for all x of the form $\frac{\pi}{2} \pm n\pi$,

$$\begin{aligned} f(10^{\frac{\pi}{2} - n\pi}) &= \cos(\log(10^{\frac{\pi}{2} - n\pi})) \\ &= \cos\left(\frac{\pi}{2} - n\pi\right) \\ &= 0 \end{aligned}$$

for all positive integers n.

19. **(C)** Let $DC = m$ and $AD = n$. By the *Pythagorean Theorem*, $AB^2 = AD^2 + DB^2$. Hence $(m + n)^2 = n^2 + 57$, which yields $m(m + 2n) = 57$. Since m and n are positive integers, the only possibilities are $m = 1$, $n = 28$ and $m = 3$, $n = 8$. The second of these gives the least possible value of $AC = m + n$, namely 11.

20. **(E)** For $n \geq 3$,

$$a_n = \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}.$$

Thus $(n-1)a_n = a_1 + a_2 + \cdots + a_{n-1}$. It follows that

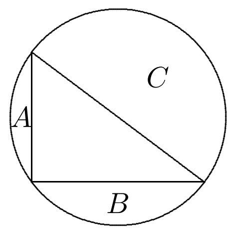
$$a_{n+1} = \frac{a_1 + a_2 + \cdots + a_{n-1} + a_n}{n} = \frac{(n-1) \cdot a_n + a_n}{n} = a_n,$$

for $n \geq 3$. Since $a_9 = 99$ and $a_1 = 19$, it follows that

$$99 = a_3 = \frac{19 + a_2}{2},$$

and hence that $a_2 = 179$. (The sequence is $19, 179, 99, 99, \ldots$.)

21. **(B)** Since $20^2 + 21^2 = 29^2$, the converse of the *Pythagorean Theorem* applies, so the triangle has a right angle. Thus its hypotenuse is a diameter of the circle, so the region with area C is a semicircle and is congruent to the semicircle formed by the other three regions. The area of the triangle is 210, hence $A + B + 210 = C$. To see that the other options are incorrect, note that



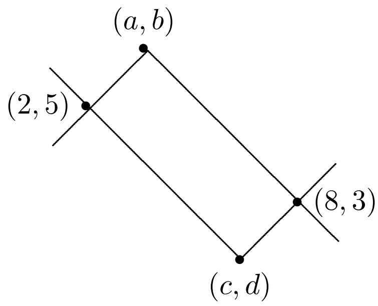
(A) $A + B < A + B + 210 = C$;

(C) $A^2 + B^2 < (A+B)^2 < (A+B+210)^2 = C^2$;

(D) $20A + 21B < 29A + 29B < 29(A + B + 210) = 29C$; and

(E) $\frac{1}{A^2} + \frac{1}{B^2} > \frac{1}{A^2} > \frac{1}{C^2}$.

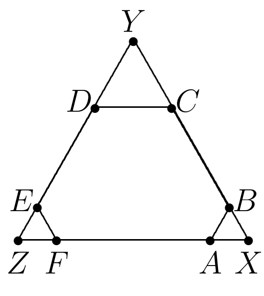
22. **(C)** The first graph is an inverted 'V-shaped' right angle with vertex at (a, b) and the second is a V-shaped right angle with vertex at (c, d). Thus $(a, b), (2, 5), (c, d)$, and $(8, 3)$ are consecutive vertices of a rectangle. The diagonals of this rectangle meet at their common midpoint, so the x-coordinate of this midpoint is $(2+8)/2 = (a+c)/2$. Thus $a+c = 10$.



OR

Use the given information to obtain the equations $5 = -|2 - a| + b$, $5 = |2 - c| + d$, $3 = -|8 - a| + b$, and $3 = |8 - c| + d$. Subtract the third from the first to eliminate b and subtract the fourth from the second to eliminate d. The two resulting equations $|8 - a| - |2 - a| = 2$ and $|2 - c| - |8 - c| = 2$ can be solved for a and c. To solve the former, first consider all $a \leq 2$, for which the equation reduces to $8 - a - (2 - a) = 2$, which has no solutions. Then consider all a in the interval $2 \leq a \leq 8$, for which the equation reduces to $8 - a - (a - 2) = 2$, which yields $a = 4$. Finally, consider all $a \geq 8$, for which the equation reduces to $a - 8 - (a - 2) = 2$, which has no solutions. The other equation can be solved similarly to show that $c = 6$. Thus $a + c = 10$.

23. **(E)** Extend $\overline{FA}$ and $\overline{CB}$ to meet at X, $\overline{BC}$ and $\overline{ED}$ to meet at Y, and $\overline{DE}$ and $\overline{AF}$ to meet at Z. The interior angles of the hexagon are 120°. Thus the triangles XYZ, ABX, CDY, and EFZ are equilateral. Since $AB = 1$, $BX = 1$. Since $CD = 2$, $CY = 2$. Thus $XY = 7$ and $YZ = 7$. Since $YD = 2$ and $DE = 4, EZ = 1$. The area of the hexagon can be found by subtracting the areas of the three small triangles from the area of the large triangle:



$$7^2\left(\frac{\sqrt{3}}{4}\right) - 1^2\left(\frac{\sqrt{3}}{4}\right) - 2^2\left(\frac{\sqrt{3}}{4}\right) - 1^2\left(\frac{\sqrt{3}}{4}\right) = \frac{43\sqrt{3}}{4}.$$

24. **(B)** Any four of the six given points determine a unique convex quadrilateral, so there are exactly $\binom{6}{4} = 15$ favorable outcomes when the chords are selected randomly. Since there are $\binom{6}{2} = 15$ chords, there are $\binom{15}{4} = 1365$ ways to pick the four chords. So the desired probability is $15/1365 = 1/91$.

25. **(B)** Multiply both sides of the equation by 7! to obtain

$$3600 = 2520a_2 + 840a_3 + 210a_4 + 42a_5 + 7a_6 + a_7.$$

It follows that $3600 - a_7$ is a multiple of 7, which implies that $a_7 = 2$. Thus,

$$\frac{3598}{7} = 514 = 360a_2 + 120a_3 + 30a_4 + 6a_5 + a_6.$$

Reason as above to show that $514 - a_6$ is a multiple of 6, which implies that $a_6 = 4$. Thus, $510/6 = 85 = 60a_2 + 20a_3 + 5a_4 + a_5$. Then it follows that $85 - a_5$ is a multiple of 5, whence $a_5 = 0$. Continue in this fashion to obtain $a_4 = 1, a_3 = 1$, and $a_2 = 1$. Thus the desired sum is $1 + 1 + 1 + 0 + 4 + 2 = 9$.

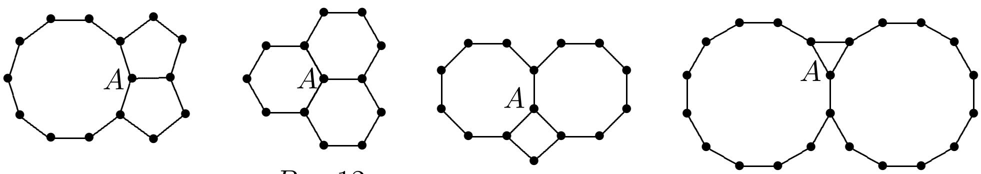
26. **(D)** The interior angle of a regular n-gon is $180(1-2/n)$. Let a be the number of sides of the congruent polygons and let b be the number of sides of the third polygon (which could be congruent to the first two polygons). Then

$$2 \cdot 180\left(1 - \frac{2}{a}\right) + 180\left(1 - \frac{2}{b}\right) = 360.$$

Clearing denominators and factoring yields the equation

$$(a-4)(b-2) = 8,$$

whose four positive integral solutions are $(a, b) = (5, 10), (6, 6), (8, 4)$, and $(12, 3)$. These four solutions give rise to polygons with perimeters of $14, 12, 14$ and 21, respectively, so the largest possible perimeter is 21.



$P = 14$ $P = 12$ $P = 14$ $P = 21$

27. **(A)** Square both sides of the equations and add the results to obtain

$$9(\sin^2 A + \cos^2 A) + 16(\sin^2 B + \cos^2 B) + 24(\sin A \cos B + \sin B \cos A) = 37.$$

Hence, $24 \sin(A+B) = 12$. Thus $\sin C = \sin(180° - A - B) = \sin(A+B) = \frac{1}{2}$, so $\angle C = 30°$ or $\angle C = 150°$. The latter is impossible because it would imply that $A < 30°$ and consequently that $3 \sin A + 4 \cos B < 3 \cdot \frac{1}{2} + 4 < 6$, a contradiction. Therefore $\angle C = 30°$.

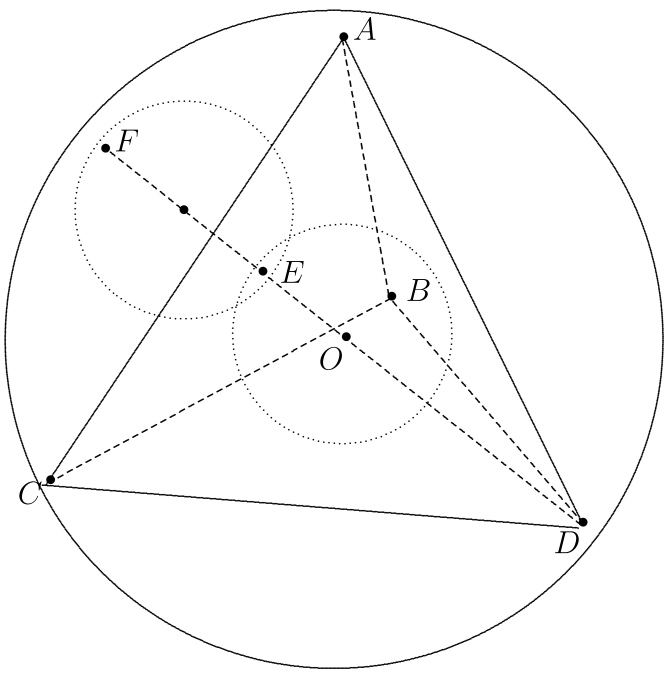
Challenge. Prove that there is a unique such triangle (up to similarity), the one for which $\cos A = \frac{5-12\sqrt{3}}{37}$ and $\cos B = \frac{66-3\sqrt{3}}{74}$.

28. **(E)** Let a, b, and c denote the number of -1's, 1's, and 2's in the sequence, respectively. We need not consider the zeros. Then a, b, c are nonnegative integers satisfying $-a + b + 2c = 19$ and $a + b + 4c = 99$. It follows that $a = 40 - c$ and $b = 59 - 3c$, where $0 \le c \le 19$ (since $b \ge 0$), so

$$x_1^3 + x_2^3 + \cdots + x_n^3 = -a + b + 8c = 19 + 6c.$$

The lower bound is achieved when $c = 0$ ($a = 40, b = 59$). The upper bound is achieved when $c = 19$ ($a = 21, b = 2$). Thus $m = 19$ and $M = 133$, so $M/m = 7$.

29. **(C)** Let A, B, C, and D be the vertices of the tetrahedron. Let O be the center of both the inscribed and circumscribed spheres. Let the inscribed sphere be tangent to the face ABC at the point E, and let its volume be V. Note that the radius of the inscribed sphere is OE and the radius of the circumscribed sphere is OD. Draw $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, and $\overline{OD}$ to obtain four congruent tetrahedra $ABCO, ABDO, ACDO$, and $BCDO$, each with volume $1/4$ that of the original tetrahedron. Because the two tetrahedra $ABCD$ and $ABCO$ share the same base, $\triangle ABC$, the ratio of the distance from O to face ABC to the distance from D to face ABC is $1/4$; that is, $OD = 3 \cdot OE$. Thus the volume of the circumscribed sphere is $27V$. Extend $\overline{DE}$ to meet the circumscribed sphere at F. Then $DF = 2 \cdot DO = 6 \cdot OE$. Thus $EF = 2 \cdot OE$, so the sphere with diameter $\overline{EF}$ is congruent to the inscribed sphere, and thus has volume V. Similarly each of the other three spheres between the tetrahedron and the circumscribed sphere have volume V. The five congruent small spheres have no volume in common and lie entirely inside the circumscribed sphere, so the ratio $5V/27V$ is the probability that a point in the circumscribed sphere also lies in one of the small spheres. The fraction $5/27$ is closer to 0.2 than it is to any of the other choices.



30. **(D)** Let $m+n=s$. Then $m^3+n^3+3mn(m+n)=s^3$. Subtracting the given equation from the latter yields

$$s^3-33^3=3mns-99mn.$$

It follows that $(s-33)(s^2+33s+33^2-3mn)=0$, hence either $s=33$ or $(m+n)^2+33(m+n)+33^2-3mn=0$. The second equation is equivalent to $(m-n)^2+(m+33)^2+(n+33)^2=0$, whose only solution, $(-33,-33)$, qualifies. On the other hand, the solutions to $m+n=33$ satisfying the required conditions are $(0,33),(1,32),(2,31),\ldots,(33,0)$, of which there are 34. Thus there are 35 solutions altogether.