Applying Poncelet's Theorem to the Pentagon and the Pentagonal Star

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1 Abstract

A special case of Poncelet's Theorem states that if all points on circle C_2 lie inside of circle C_1 and if a convex n-polygon, $n \geq 3$, or an n-star, $n \geq 5$, is inscribed in circle C_1 and circumscribed about circle C_2, then there exists a family of such n-polygons and n-stars.

Suppose all points on C_2 lie inside of $C_1, R, r,$ are the radii of C_1, C_2 respectively and ρ is the distance between the centers of C_1, C_2. For $n \geq 3$, in a companion paper we give an algorithm that computes the necessary and sufficient conditions on R, r, ρ, where $R > r + \rho, r > 0$, so that if we start at any arbitrary point Q on C_1 and draw successive tangents to C_2 (counterclockwise about the center of C_2) then we will return to Q in exactly n steps and not return to Q in fewer than n steps. This will create the above family of n-polygons and n-stars.

However, when $n \geq 5$, this companion paper relies on computers to find these conditions. In some ways, this is a sign of defeat. In this paper, we illustrate for $n = 5$ a technique that can compute these exact same necessary and sufficient conditions on R, r, ρ without using a computer.

2 Introduction

A special case of Poncelet's Theorem states that if all points on circle C_2 lie inside of circle C_1 and if a convex n-polygon, $n \geq 3$, or an n-star, $n \geq 5$, is inscribed in circle C_1 and circumscribed about circle C_2 then there exists a family of such n-polygons and n-stars. Suppose all points on C_2 lie inside of C_1, R, r are the radii of C_1, C_2 respectively and ρ is the distance between the centers of C_1, C_2.

For $n = 5$, we illustrate a technique that can be carried out by hand that computes the necessary and sufficient conditions on R, r, ρ, where $R > r + \rho, r > 0$, so that if we start at any point Q on C_1, and draw successive tangents to C_2 (counterclockwise about the center of C_2) then we will return to Q in exactly 5 steps and not return to Q in fewer than 5 steps.

If we consider $R > \rho \geq 0$ to be arbitrary but fixed and consider $r > 0$ to be a variable, then we end up with two polynomial equations $P(R, \rho, r) = 0, \overline{P}(R, \rho, r) = 0$ that are each of third degree in the variable r. Each of the equations $P(R, \rho, r) = 0, \overline{P}(R, \rho, r) = 0$ has exactly one r-root that satisfies $R > r + \rho, r > 0$. This r-root of $P(R, \rho, r) = 0$ is the value of r so that we get a family of pentagonal stars and this r-root of $\overline{P}(R, \rho, r) = 0$ is the value of r so that we get a family of pentagons when we start at any arbitrary point Q on C_1.

In this paper, we only deal with the 5-star. The geometric reasoning for the convex pentagon is very similar. Also, we know from the companion paper that the two polynomials $P(R, \rho, r), \overline{P}(R, \rho, r)$ are related by $\overline{P}(R, \rho, r) = P(-R, \rho, r) = P(R, \rho, -r)$. Thus, we can immediately write the polynomial $\overline{P}(R, \rho, r)$ directly from the polynomial $P(R, \rho, r)$ without doing any additional work.

3 A Preliminary Unfactored Form of the Polynomial $P(R, \rho, r)$

In this section, for the pentagonal star, we compute a preliminary first version called $P^*(R, \rho, r)$ of the polynomial $P(R, \rho, r)$. Then in Section 4, we refine $P^*(R, \rho, r)$ by factoring it into the following four irreducible factors where $R^2 - p^2 = \theta$.

$P^*(R, \rho, r) = [8\rho^2 Rr^3 - 4R^2\theta r^2 - 2R\theta^2 r + \theta^3] \cdot [2Rr + \theta] \cdot [r - R + \rho]^2$ and we call $P(R, \rho, r) = 8\rho^2 Rr^3 - 4R^2\theta r^2 - 2R\theta^2 r + \theta^3$.

Of course, for the pentagon we have $\overline{P}(R, \rho, r) = P(-R, \rho, r) = P(R, \rho, -r) = -8\rho^2 Rr^3 - 4R^2\theta r^2 + 2R\theta^2 r + \theta^3$.

The linear factor $r-R+\rho=0$ in $P^*(R,\rho,0)$ is extraneous since we require $R>r+\rho, r>0$. Also, the factor $2Rr+\theta=0$ in $P^*(R,\rho,r)$ is an Euler type of equation which has only an extraneous negative r-root since $\theta>0$.

By Poncelet's Theorem, we can use any drawing to compute $P^*(R,\rho,r)=0$ that simplifies the problem. Therefore, by Poncelet's Theorem, the simple drawing of Fig. 1 is all that we need to compute $P^*(R,\rho,r)=0$ for the pentagonal star. An analogous drawing is used for the convex pentagon. The θ in figure 1 is different from the θ is $\theta=R^2-\rho^2$.

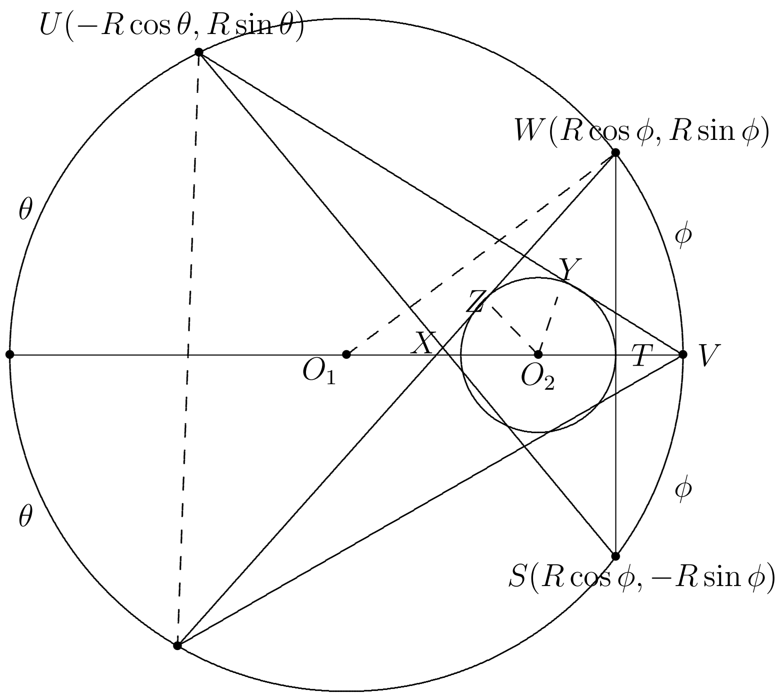

Fig. 1 A Drawing to Compute $P^*(R,\rho,r)=0$

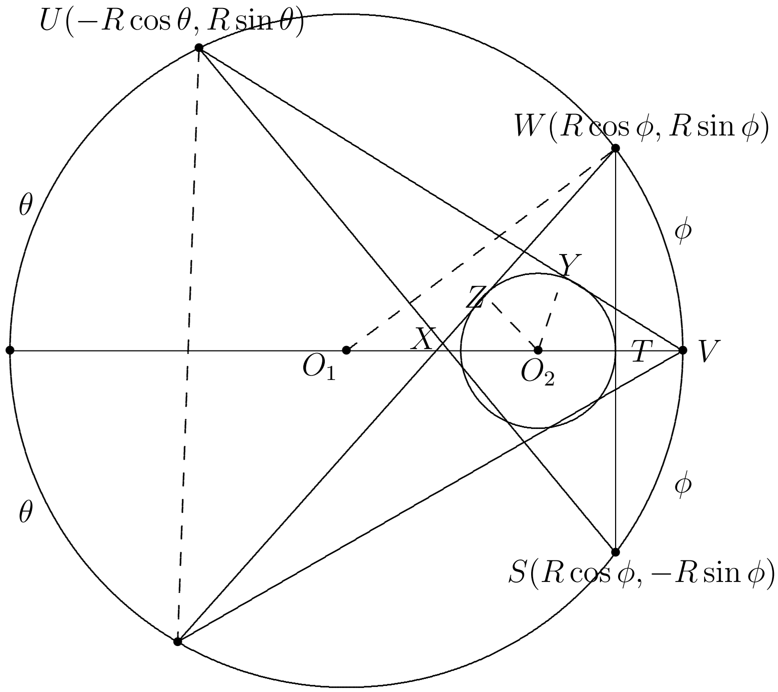

Fig. 1 A Drawing to Compute $P^*(R,\rho,r)=0$

O_1 is the center of the big circle C_1 and O_2 is the center of the inside circle C_2. R, r are the radii of C_1, C_2 respectively and $\rho = O_1O_2$ is the distance between the centers O_1 and O_2.

We immediately have (a) and (b).

(a) $O_2y = (O_2v)\cdot\sin\frac{\theta}{2} = (R-\rho)\sin\frac{\theta}{2} = r.$

(b) $O_1t = (O_1w)\cdot\cos\phi = R\cos\phi = \rho + r.$

The parametric equation of the line US is $x = R\cos\phi - t\left(R\cos\phi + R\cos\theta\right), y = -R\sin\phi + t\left(R\sin\phi + R\sin\theta\right)$ where $t \in R$.

Therefore, $x\left(\sin\phi+\sin\theta\right)+y\left(\cos\phi+\cos\theta\right) = R\cos\phi\left(\sin\phi+\sin\theta\right)-R\sin\phi\left(\cos\phi+\cos\theta\right) = R\sin\theta\cos\phi - R\cos\theta\sin\phi = R\sin\left(\theta-\phi\right) = 2R\sin\left(\frac{\theta-\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right).$

Therefore, $2x\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)+2y\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)=2R\sin\left(\frac{\theta-\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right).$

Therefore, $x\sin\frac{\theta+\phi}{2}+y\cos\frac{\theta+\phi}{2}=R\sin\frac{\theta-\phi}{2}$ is the equation of the line US.

Letting $y=0$ in this equation of the line US, we have $O_1x=\frac{R\sin\frac{\theta-\phi}{2}}{\sin\frac{\theta+\phi}{2}}$.

Therefore, $xO_2=O_1O_2-O_1x=\rho-O_1x=\rho-\frac{R\sin\frac{\theta-\phi}{2}}{\sin\frac{\theta+\phi}{2}}$. Also, $O_2z=xO_2\cdot\sin\frac{\theta+\phi}{2}=r=\left[\rho-\frac{R\sin\frac{\theta-\phi}{2}}{\sin\frac{\theta+\phi}{2}}\right]\sin\frac{\theta+\phi}{2}$. Therefore, $(*)\,.\rho\sin\frac{\theta+\phi}{2}-R\sin\frac{\theta-\phi}{2}=r$.

Now $\sin^2\frac{\theta+\phi}{2}=\frac{1-\cos(\theta+\phi)}{2}$. Also $\sin^2\frac{\theta-\phi}{2}=\frac{1-\cos(\theta-\phi)}{2}$. Also $\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}=\frac{1}{2}\cos\phi-\frac{1}{2}\cos\theta$.

Squaring $(*)$ and making these substitutions we have $\rho^2\left[1-\cos\left(\theta+\phi\right)\right]+R^2\left[1-\cos\left(\theta-\phi\right)\right]-2\rho R\left[\cos\phi-\cos\theta\right]=2r^2$.

Therefore, $-R^2\cos\left(\theta-\phi\right)-\rho^2\cos\left(\theta+\phi\right)+2\rho R\left[\cos\theta-\cos\phi\right]=2r^2-R^2-\rho^2$.

Therefore, $-R^2\left[\cos\theta\cos\phi+\sin\theta\sin\phi\right]-\rho^2\left[\cos\theta\cos\phi-\sin\theta\sin\phi\right]+2\rho R\left[\cos\theta-\cos\phi\right]=2r^2-R^2-\rho^2$.

Therefore, $\left(-R^2+\rho^2\right)\sin\theta\sin\phi=2r^2-R^2-\rho^2-2\rho R\left(\cos\theta-\cos\phi\right)+\left(R^2+\rho^2\right)\left(\cos\theta\cos\phi\right).$

Squaring we have

$$\begin{aligned}
&\left(-R^2+\rho^2\right)^2\left(1-\cos^2\theta\right)\left(1-\cos^2\phi\right)\\
&= (**)\left(-R^2+\rho^2\right)^2\left(1-\cos\theta\right)\left(1+\cos\theta\right)\left(1-\cos\phi\right)\left(1+\cos\phi\right)\\
&= \left[2r^2-R^2-\rho^2-2\rho R\left(\cos\theta-\cos\phi\right)+\left(R^2+\rho^2\right)\left(\cos\theta\cos\phi\right)\right]^2.
\end{aligned}$$

Since we have a homogeneous geometric equation in the variables R, r, ρ, it is convenient to let $R=1$.

From (a), (b) we know that $\sin\frac{\theta}{2}=\frac{r}{R-\rho}=\frac{r}{1-\rho}$ and $\cos\phi=\frac{\rho+r}{R}=\rho+r$.

Therefore, $\cos\theta=1-2\sin^2\frac{\theta}{2}=1-2\left(\frac{r}{1-\rho}\right)^2$ and $\cos\phi=\rho+r$. Therefore, $1-\cos\theta=2\left(\frac{r}{1-\rho}\right)^2, 1+\cos\theta=2-2\left(\frac{r}{1-\rho}\right)^2=\frac{2(1-\rho)^2-2r^2}{(1-\rho)^2}, 1-\cos\phi=1-\rho-r, 1+\cos\phi=1+\rho+$

$r, \cos\theta - \cos\phi = 1 - \rho - r - 2\left(\frac{r}{1-\rho}\right)^2$ and $\cos\theta\cos\phi = (\rho + r)\left(1 - 2\left(\frac{r}{1-\rho}\right)^2\right)$.

If we make these substitutions and also substitute $R = 1$ in $(**)$ and multiply the equation by $(1-\rho)^4$ and partially simplify the equation by straightforward calculations and also transpose everything to one side of the equation, then we have the following equation which we call the preliminary polynomial equation.

$P^*(R, r, \rho) = \left[\left[2r^2 + 2\rho r + \rho^2 - 2\rho - 1\right](1-\rho)^2 + 4\rho r^2 + (1+\rho^2)\left[(1-\rho)^2 - 2r^2\right](\rho + r)\right]^2 - 4(1-\rho^2)^2\left[r^2(1-\rho-r)^2(1-\rho+r)(1+\rho+r)\right] = 0.$

4 Factoring the Preliminary Equation $P^*(R, r, \rho) = 0$ of Section 3 into Irreducible Factors

The above preliminary polynomial equation $P^*(R, r, \rho) = P^*(1, r, \rho) = 0$ in the variable r at first glance appears to be intractable. However, if we substitute specific values of ρ e.g. $\rho = 0, \rho = 1, \rho = 2$ we quickly conjecture that this polynomial equation can probably be factored into simple factors.

If we substitute $\rho = 0$, the preliminary equation becomes $P^*(R, \rho, r) = P^*(1, 0, r) = \left[2r^2 - 1 + (1-2r^2)r\right]^2 - 4r^2(1-r^2)^2 = 0$ which is equivalent to $\left[2r^3 - 2r^2 - r + 1\right]^2 - \left[2r^3 - 2r\right]^2 = \left[-2r^2 + r + 1\right]\left[4r^3 - 2r^2 - 3r + 1\right] = -(r-1)(2r+1)(r-1)(4r^2 + 2r - 1) = -(r-1)^2(2r+1)(4r^2 + 2r - 1) = 0$.

By making other substitutions for ρ, we soon conjecture that $P^*(R, r, \rho) = P^*(1, r, \rho) = P(1, r, \rho)(2r + 1 - \rho^2)(r - 1 + \rho)^2 = 0$ where $P(1, r, \rho)$ is a 3rd degree polynomial in r.

We now rigorously prove this conjecture. By direct substitution of $r = 1 - \rho$ into

$P^*(1,r,\rho)$ we can easily prove that $r = 1-\rho$ is a double r-root of $P^*(1,r,\rho) = 0$. To see this, we see that $P^*(1,r,\rho)$ is of the form $P^* = [xxx]^2 - [yyy](1-\rho-r)^2$ and we only need to show that $[xxx] = 0$ when $r = 1-\rho$ to show that $r = 1-\rho$ is a double root of $P^*(1,r,\rho) = 0$

Now in $[xxx]$ when $r = 1-\rho$ we see that $2r^2 + 2\rho r + \rho^2 - 2\rho - 1 = 2r(\rho + r) + \rho^2 - 2\rho - 1 = 2(1-\rho) + \rho^2 - 2\rho - 1 = \rho^2 - 4\rho + 1$. Therefore, in $[xxx]$ when $r = 1-\rho$ we have $[2r^2 + 2\rho r + \rho^2 - 2\rho - 1][1-\rho]^2 + 4\rho r^2 = (\rho^2 - 4\rho + 1)(1-\rho)^2 + 4\rho(1-\rho)^2 = (\rho^2+1)(1-\rho)^2$.

Also, in $[xxx]$ when $r = 1-\rho$, we have

$$\left(1+\rho^2\right)\left[(1-\rho)^2 - 2r^2\right](\rho + r) = \left(1+\rho^2\right)\left[(1-\rho)^2 - 2(1-\rho)^2\right] = -\left(1+\rho^2\right)(1-\rho)^2.$$

Therefore, when $r = 1-\rho, [xxx] = (\rho^2+1)(1-\rho)^2 - (1+\rho^2)(1-\rho)^2 = 0$. Therefore, $r = 1-\rho$ is a double r-root of $P^*(1,r,\rho) = 0$.

The proof that $2r + 1 - \rho^2 = 0$ gives an r-root of $P^*(1,r,\rho) = 0$ takes a little longer but it is completely straightforward.

Therefore, we know that $P^*(1,r,\rho) = (ar^3 + br^2 + cr + d)(2r + 1 - \rho^2)(r - 1 + \rho)^2$ where a, b, c, d need to be determined.

Now, $P^*(1,r,\rho) = a_0 r^6 + a_1 r^5 + a_2 r^4 + a_3 r^3 + a_4 r^2 + a_5 r + a_6$.

For $P^*(1,r,\rho)$ it is fairly easy by straightforward calculations to compute the following coefficients.

$a_0 = 16\rho^2$.

$a_1 = -8(1-\rho)(-\rho^3 + 3\rho^2 + \rho + 1)$.

$a_5 = -2(1-\rho)^5(1+\rho)^4$.

$a_6 = (1-\rho)^6 (1+\rho)^4$.

As an example, to compute a_0 we have $a_0 = 4(1+\rho^2)^2 - 4(1-\rho^2)^2 = 16\rho^2$.

Also, to compute a_5 we have the following relevant terms,

$$\left[\begin{array}{l} 2\rho(1-\rho)^2 r + (\rho^2-2\rho-1)(1-\rho)^2 + \\ (1+\rho^2)(1-\rho)^2 r + (1+\rho^2)(1-\rho)^2 \rho \end{array}\right]^2 = \left[\begin{array}{l} 2\rho(1-\rho)^2 r + (1+\rho^2)(1-\rho)^2 r + \\ (\rho^2-2\rho-1)(1-\rho)^2 + (1+\rho^2)(1-\rho)^2 \rho \end{array}\right]^2 =$$

$$\left[\begin{array}{l} (1+\rho)^2(1-\rho)^2 r + \\ (\rho^3+\rho^2-\rho-1)(1-\rho)^2 \end{array}\right]^2 = \left[(1+\rho)^2(1-\rho)^2 r - (1+\rho)^2(1-\rho)^3\right]^2.$$

From this, we see that $a_5 = -2(1-\rho)^5(1+\rho)^4$. To compute a_6 we let $r=0$ in $P^*(1,r,\rho)$ and we have $a_6 = \left[(\rho^2-2\rho-1)(1-\rho)^2 + (1+\rho^2)(1-\rho)^2\rho\right]^2 = (1-\rho)^4\left[\rho^3+\rho^2-\rho-1\right]^2 = (1-\rho)^4\left[(\rho+1)^2(\rho-1)\right]^2 = (1-\rho)^6(1+\rho)^4$.

The calculation of a_1 is a little longer but it is completely straightforward. However, we must be careful not to overlook anything in computing a_1. Once we know a_0, a_1, a_5, a_6, it is completely straight forward to compute $P(1,r,\rho) = ar^3+br^2+cr+d = 8\rho^2r^3 - 4\theta r^2 - 2\theta^2 r + \theta^3$ where $\theta = 1-\rho^2$. So $P^*(1,r,\rho) = P(1,r,\rho)\cdot(2r+1-\rho^2)(r-1+\rho)^2$. We now proceed to rigorously prove this. We first note that $(8\rho^2r^3 - 4\theta r^2 - 2\theta^2 r + \theta^3)(2r+\theta) = 16\rho^2r^4 - 8\theta^2r^3 - 8\theta^2r^2 + \theta^4$. Therefore, we prove that $P^*(1,r,\rho)$

$= (16\rho^2r^4 - 8\theta^2r^3 - 8\theta^2r^2 + \theta^4)(r-1+\rho)^2 = (16\rho^2r^4 - 8\theta^2r^3 - 8\theta^2r^2 + \theta^4)\left(r^2 - 2(1-\rho)r + (1-\rho)^2\right)$.

This equality will be true if and only if the equality correctly computes the above values for a_0, a_1, a_5, a_6, since we have already proved that $2r+\theta$ and $(r-1+\rho)^2$ are factors of $P^*(1,r,\rho)$. Now $a_0 = 16\rho^2$ is obviously computed correctly.

Also, $a_1 = -32\rho^2(1-\rho) - 8(1-\rho^2)^2 =$

$-8(1-\rho)\left[4\rho^2 + (1+\rho)^2(1-\rho)\right] = -8(1-\rho)\left[-\rho^3+3\rho^2+\rho+1\right]$.

Also, $a_5 = -2(1-\rho)(1-\rho^2)^4 = -2(1-\rho)^5(1+\rho)^4$.

Also, $a_6 = \left(1-\rho^2\right)^4 \left(1-\rho\right)^2 = \left(1-\rho\right)^6 \left(1+\rho\right)^4$.

Therefore, we have now rigorously proved that $P^*\left(1,r,\rho\right) = P\left(1,r,\rho\right) \cdot \left(2r+\theta\right)\left(r-1+\rho\right)^2 = \left(8\rho^2 r^3 - 4\theta r^2 - 2\theta^2 r + \theta^3\right)\left(2r+\theta\right)\left(r-1+\rho\right)^2$ where $\theta = 1-\rho^2$.

Of course, this equation can be written for $P^*\left(R,r,\rho\right)$ in the three variables R, r, ρ since the equation is a homogeneous geometric equation. This equation $P\left(R,r,\rho\right) = P\left(1,r,\rho\right)$ is exactly the same equation that we derived in a companion paper by using a computer. This computer derivation was carried out independently by Prof. Benjamin Klein of Davidson College and by Parker Garrison. So we now have three independent verifications of this one equation.

5 Studying $P\left(R,r,\rho\right) = P\left(1,r,\rho\right)$

If $R = 1 > \rho \geq 0$, we require $R = 1 > r+\rho, r > 0$.

It is easy to show that $P\left(1,r,\rho\right) = 8\rho^2 r^3 - 4\theta r^2 - 2\theta^2 r + \theta^3$ is irreducible in the rational field.

Letting $R = 1, 0 < \rho < 1$, we know by Descarte's law of signs that $P\left(1,r,\rho\right) = 0$ has two or zero positive r-roots for each fixed $0 < \rho < 1$. For each fixed $0 < \rho < 1$ we show that $P\left(1,r,\rho\right) = 0$ has one r-root that satisfies $0 < r < 1-\rho$. ($\rho = 0$ is easy to deal with.)

Now $P\left(1,r,\rho\right) = P\left(1,+\infty,\rho\right) > 0$.

Also, $P\left(1,r,\rho\right) = P\left(1,0,\rho\right) > 0$. If we show that $P\left(1,r,\rho\right) = P\left(1,1-\rho,\rho\right) < 0$, then it will follow that for each fixed $0 < \rho < 1, P\left(1,r,\rho\right) = 0$ will have one r-root that satisfies $0 < r < 1-\rho$.

Now $P(1,r,\rho) = P(1,1-\rho,\rho) < 0$ if and only if

$$(1-\rho)^3\left[8\rho^2 - 4(1+\rho) - 2(1+\rho)^2 + (1+\rho)^3\right] < 0.$$

This is true if and only if

$$(1-\rho)^3\left[-4\left(1+\rho-2\rho^2\right) - (1+\rho)^2(2-(1+\rho))\right] = (1-\rho)^3\left[-4(1+2\rho)(1-\rho) - (1+\rho)^2(1-\rho)\right] < 0,$$

which is true.

Therefore, for each $R = 1 > \rho \geq 0$, we see that $P(1,r,\rho) = 0$ has one r-root that satisfies $R = 1 > r+\rho, r > 0$.

If $R = 1 > \rho \geq 0$ are fixed, this r-root is the radius of the inside circle C_2 so that we have a family of 5-stars that are inscribed in C_1 and circumscribed about C_2 when the distance between the centers of C_1, C_2 is $p, R = 1$ is the radius of C_1 and r is the radius of C_2.

6 Extending the Equation to Include Convex Pentagons

By using analogous reasoning we can show that the companion equation $\overline{P}(R,r,\rho) = P(-R,r,\rho) = P(R,-r,\rho) = -8\rho^2Rr^3 - 4R^2\theta r^2 + 2R\theta^2 r + \theta^3 = 0$, where $\theta = R^2 - \rho^2$, is the relation between R, r, ρ where $R > r+\rho, r > 0$, so that we have a family of convex pentagons that are inscribed in C_1 and circumscribed about C_2. From the companion paper, we know that the equation $\overline{P}(R,r,\rho) = 0$ for the convex pentagon can be written directly from $\overline{P}(R,r,\rho) = P(-R,r,\rho) = P(R,-r,\rho)$. It is easy to show that for $R = 1 > \rho \geq 0$, there exists exactly one real r-root of $\overline{P}(1,r,\rho) = 0$ that satisfies $R = 1 > r+\rho, r > 0$.

7 Concluding Remarks

Everything in this paper was done completely by hand and this adds completeness to a computer only derived solution. The advantage of the computer derived solution is that it is less mentally demanding and requires less thought to carry out.

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