Classifying Similar Triangles Inscribed in a Given Triangle

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1 Abstract

Suppose $\triangle ABC$ and $\triangle A'B'C'$ are given triangles whose vertices A, B, C and A', B', C' are given in counterclockwise order. We show how to find and classify all triangles $\triangle \overline{ABC}$ inscribed in $\triangle ABC$ with $\overline{A}, \overline{B}, \overline{C}$ lying on the infinite lines BC, CA, AB respectively and with angle $\overline{A} = A', \overline{B} = B', \overline{C} = C'$ so that $\triangle \overline{ABC}$ is directly (or inversely) similar to $\triangle A'B'C'$. This seemingly complex problem has a remarkably simple and revealing solution.

2 Introduction

Suppose $\triangle ABC$ and $\triangle A'B'C'$ are similar triangles with angle $A = A', B = B', C = C'$. If the rotation determined by the points A, B, C taken in order is counterclockwise and the rotation A', B', C' is counterclockwise, or vice versa, the triangles are said to be directly similar. If one rotation is clockwise and the other rotation is counterclockwise, the triangles are said to be inversely similar. See [1], p.37. If one vertex A of a variable triangle $\triangle ABC$ is fixed, a second vertex B describes a given straight line l and the triangle remains directly (or inversely) similar to a given triangle, then the third vertex C describes a straight line $\bar{l}$. See [1], p.49.

Figure 1 makes this easy to understand.

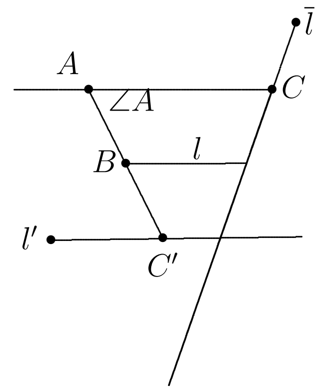

Figure 1. $\triangle ABC$ is directly similar to a given $\triangle$.

As B varies along the given line l, define C' so that $AC' = AC$ as shown in Fig. 1. It is obvious that C' lies on a straight line l' that is parallel to l since $\frac{AC}{AB} = \frac{AC'}{AB}$ is fixed. The point C is obtained from the point C' by rotating AC' by $\angle A$ radians about the axis A as shown in Fig. 1. Therefore, it is easy to see that point C lies on the line $\bar{l}$ that is obtained by rotating line l' by $\angle A$ radians about the axis A. A simple formal proof is given on p. 49, [1].

Suppose $\triangle ABC, \triangle A'B'C'$ are given triangles with vertices A, B, C and A', B', C' given in counterclockwise order. Also, suppose $\overline{A}$ is any fixed point on side BC of $\triangle ABC$. If we now let $\overline{B}$ vary along side CA, it is easy to see from the above that there exists a unique $\triangle\overline{ABC}$ inscribed in $\triangle ABC$ with $\overline{A}$ the fixed point, $\overline{B}$ lying on CA and $\overline{C}$ lying on AB so that $\triangle\overline{ABC}$ is directly (or inversely) similar to $\triangle A'B'C'$ with angle $\overline{A} = A', \overline{B} = B', \overline{C} = C'$.

If we vary $\overline{A}$ on BC and study the action as the above $\triangle\overline{ABC}$ varies, physical reasoning will sometimes suggest that exactly one of these triangles $\triangle\overline{ABC}$ will have the following property. If we draw the perpendiculars to sides BC, CA, AB at the points $\overline{A}, \overline{B}, \overline{C}$ then these three perpendiculars will be concurrent. In Section 3 we give a rigorous geometric proof of this fact.

In the rest of the paper, we solve and discuss the following Main Problem.

Main Problem 1 Suppose $\triangle ABC, \triangle A'B'C'$ are given triangles with vertices A, B, C and A', B', C' listed in counterclockwise order. Show how to classify in a revealing way all triangles $\triangle\overline{ABC}$ inscribed in $\triangle ABC$ with $\overline{A}, \overline{B}, \overline{C}$ lying on the infinite sides BC, CA, AB respectively so that each $\triangle\overline{ABC}$ is directly similar to $\triangle A'B'C'$ with angle $\overline{A} = A', \overline{B} = B', \overline{C} = C'$.

In Section 5, by considering inverse points P, P' of the circumcircle, of $\triangle ABC$, we can also allow $\triangle\overline{ABC}, \triangle A'B'C'$ to be inversely similar.

3 Initial Work on Solving the Main Problem

The triangle formed by the feet of the perpendiculars of a given point P upon the sides of a given triangle $\triangle ABC$ is called the pedal triangle of P with respect to $\triangle ABC$. p.173, [1].

Problem 2 Suppose $\triangle ABC, \triangle A'B'C'$ are given triangles with vertices A, B, C and A', B', C' listed in counterclockwise order. We wish to geometrically construct a point P so that if $\triangle \overline{ABC}$ is the pedal triangle of P with respect to $\triangle ABC$ with $\overline{A}, \overline{B}, \overline{C}$ lying on sides BC, CA, AB respectively then $\triangle \overline{ABC}$ is directly similar to $\triangle A'B'C'$ with angle $\overline{A} = A', \overline{B} = B', \overline{C} = C'$. In Section 5 we show that this point P is also unique.

Note The discussion of inverse points P, P' of the circumcircle of $\triangle ABC$ in Section 5 will allow us to use the solution to Problem 2 to immediately construct a point P so that the triangle $\triangle \overline{ABC}$ of Problem 2 is inversely similar to the given $\triangle A'B'C'$.

Finally in our solution to Problem 2 we mention the technical matter that P must lie inside of the circumcircle of $\triangle ABC$ in order for the vertices of the pedal $\triangle \overline{ABC}$ of P to be counterclockwise. We prove this fact in Section 5 by using a continuity argument with the Simpson lines.

Solution, Problem 2 In triangles $\triangle ABC, \triangle A'B'C'$ we have angles A, B, C and A', B', C'. We define the following three angles: $A + A' = \angle A + \angle A', \angle B + \angle B', \angle C + \angle C'$.

It is obvious that $\angle A + \angle A' > \angle A, \angle B + \angle B' > \angle B, \angle C + \angle C' > \angle C$.

Also, since $A + A' + B + B' + C + C' = 360$, it is obvious that at most one of the three inequalities $A + A' \geq 180°, B + B' \geq 180°, C + C' \geq 180°$ can be true. Therefore, at least two of the following three inequalities must be true. $A + A' < 180°, B + B' < 180°, C + C' < 180°$.

By symmetry, let us suppose that $A < A + A' < 180°, C < C + C' < 180°$.

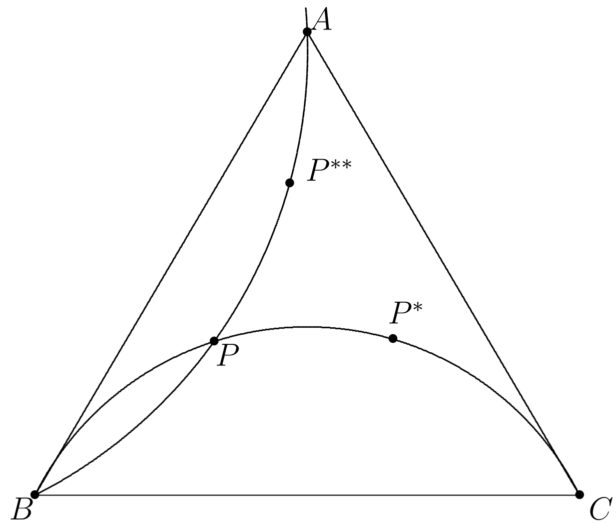

Fig. 2. $\angle BP^*C = A + A', \angle AP^{**}B = C + C'$.

A standard locus (p.12, [1]) is that the locus of a point P on one side of a given line segment at which this line segment subtends a given angle is an arc of a circle passing through the ends of the line segment. In Fig. 2 let us define the locus of points P^*, P^{**} such that $\angle BP^*C = A + A'$ and

$\angle AP^{**}B = C + C'$.

Since $A + A' < 180°, C + C' < 180°$, these two loci are the arcs of the two circles shown in Fig. 2. Also, since $A < A + A', C < C + C'$, we know that these two arcs lie inside of the circumcircle of $\triangle ABC$.

In Fig. 2 we note that $\angle BCP^*$ approaches 0 as P^* approaches B. Therefore, as P^* approaches B we see that $\angle P^*BC$ approaches $180° - \angle BP^*C = 180° - A - A'$. Likewise, as P^{**} approaches B we see that $\angle P^{**}BA$ approaches $180° - C - C'$. Therefore, as both P^*, P^{**} approach B, we see that $\angle P^*BC + \angle P^{**}BA$ approaches $(180° - A - A') + (180° - C - C') = (180° - A - C) + (180° - A' - C') = B + B'$. Since $B + B' > B$, we see that the two arcs shown in Fig. 2 will interlock precisely as we have shown in Fig. 2. Therefore, since both arcs also lie inside of the circumcircle of $\triangle ABC$ will know that there is a point P where $P \neq B$ such that P is the intersection of these two arcs.

Of course, P lies inside of the circumcircle of $\triangle ABC$. However, P can lie outside of $\triangle ABC$ or lie on line segment AC. Also, we know that $\angle BPC = A + A', \angle APB = C + C'$. Let $\triangle \overline{ABC}$ be the pedal triangle of P with respect to $\triangle ABC$. Since P lies inside of the circumcircle of $\triangle ABC$, we know that the vertices $\overline{A}, \overline{B}, \overline{C}$ of $\triangle \overline{ABC}$ are listed in counterclockwise order since vertices A, B, C of $\triangle ABC$ are listed in counterclockwise order.

We now show that $\triangle \overline{ABC}$ is directly similar to the given $\triangle A'B'C'$.

In Fig. 3 we draw the case where P lies inside of $\triangle ABC$. The proof when P lies outside of $\triangle ABC$ or on side CA is almost exactly the same, and we have carefully checked this. We have also checked other possibilities, such as the case where $\triangle ABC$ is obtuse, etc.

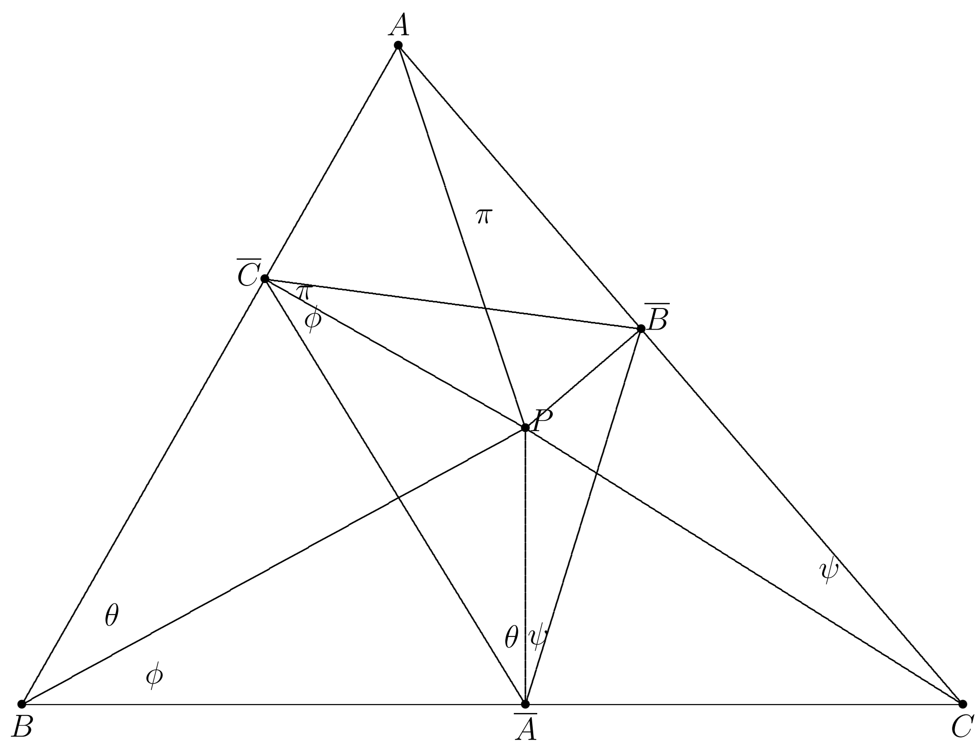

Fig.3. Proving angles $\overline{A} = A', \overline{B} = B', \overline{C} = C'$

In Fig. 3 we note that the points $\overline{A}, B, \overline{C}, P$ are concyclic (i.e., lie on a circle), points $\overline{B}, C, \overline{A}, P$ are concyclic and points $\overline{C}, A, \overline{B}, P$ are concyclic since $P\overline{A} \perp BC, P\overline{B} \perp CA$ and $P\overline{C} \perp AB$. Therefore, we have the equality of the angles $\theta = \theta, \phi = \phi, \psi = \psi, \pi = \pi$ shown in Fig. 3 since angles inscribed inside of the same arc of a circle are equal.

Now, $\angle\overline{A} = \overline{A} = \theta + \psi$. Also,

$$\begin{aligned}\angle BPC &= A + A' = 180^\circ - \angle PBC - \angle PCB = 180^\circ - (B - \theta) - (C - \psi) \\ &= 180^\circ - B - C + \theta + \psi = A + \theta + \psi.\end{aligned}$$

Therefore, $A + A' = A + \theta + \psi$ implies $A' = \theta + \psi = \overline{A}$.

Likewise, $\angle\overline{C} = \overline{C} = \phi + \pi$. Also

$$\begin{aligned}\angle APB &= C + C' = 180^\circ - \angle PAB - \angle PBA \\ &= 180^\circ - (A - \pi) - (B - \phi) \\ &= 180^\circ - A - B + \pi + \phi = C + \pi + \phi\end{aligned}$$

Therefore, $C + C' = C + \pi + \phi$ implies $C' = \pi + \phi = \overline{C}$. Since $\overline{A} = A', \overline{C} = C'$ we know that $\overline{B} = B'$. ∎

4 Solving the Main Problem

Triangles $\triangle ABC, \triangle A'B'C'$ are given with A, B, C and A', B', C' listed in counterclockwise order. From Section 3 we know how to find a point P such that if $\triangle\overline{ABC}$ is the pedal triangle of P with respect to $\triangle ABC$ with $\overline{A}, \overline{B}, \overline{C}$ lying on sides BC, CA, AB respectively, then $\triangle\overline{ABC}$ will be directly similar to $\triangle A'B'C'$ with angle $\overline{A} = A', \overline{B} = B', \overline{C} = C'$

Let us now turn (i.e., rotate) the line segments $P\overline{A}, P\overline{B}, P\overline{C}$ by θ radians about the axis P to define new points $\overline{A}', \overline{B}', \overline{C}'$ on sides BC, CA, AB as shown in Fig. 4.

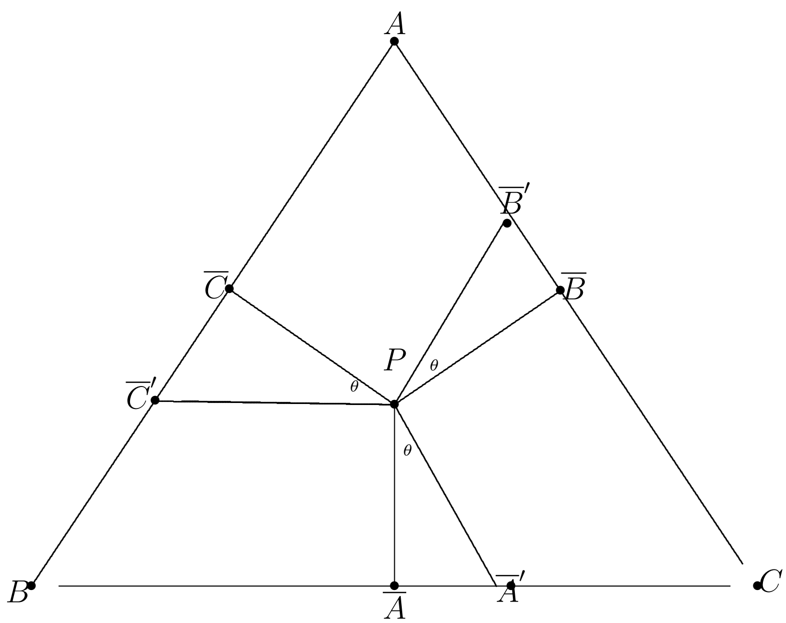

Fig.4. Turning $\triangle\overline{ABC}$ by θ radians

Now $\frac{P\overline{A}'}{P\overline{A}} = \frac{P\overline{B}'}{P\overline{B}} = \frac{P\overline{C}'}{P\overline{C}} = \frac{1}{\cos\theta} = \sec\theta$.
Since the line segments $P\overline{A}, P\overline{B}, P\overline{C}$ have each been rotated by θ radians and since

$\frac{P\overline{A}'}{P\overline{A}} = \frac{P\overline{B}'}{P\overline{B}} = \frac{P\overline{C}'}{P\overline{C}}$ we see that $\triangle\overline{A}'\overline{B}'\overline{C}'$ is directly similar to $\overline{ABC}$. Therefore, $\triangle\overline{A}'\overline{B}'\overline{C}'$ is directly similar to the given $\triangle A'B'C'$.

From Section 2, we know that for every fixed $\overline{A}'$ on side BC there is a unique triangle $\triangle\overline{A}'\overline{B}'\overline{C}'$ with $\overline{A}'$ the fixed point, $\overline{B}'$ lying on CA and $\overline{C}'$ lying on AB with angles $\overline{A}' = A', \overline{B}' = B', \overline{C}' = C'$ such that $\triangle\overline{A}'\overline{B}'\overline{C}'$ is directly similar to $\triangle A'B'C'$. Therefore, we have now easily constructed all triangles $\triangle\overline{A}'\overline{B}'\overline{C}'$ with $\overline{A}', \overline{B}', \overline{C}'$ lying on sides BC, CA, AB, with angles $\overline{A}' = A', \overline{B}' = B', \overline{C}' = C'$ such that $\triangle\overline{A}'\overline{B}'\overline{C}'$ is directly similar to $\triangle A'B'C'$.

5 Comments on the Solution of Section 4

Since $\frac{P\overline{A}'}{P\overline{A}} = \frac{P\overline{B}'}{P\overline{B}} = \frac{P\overline{C}'}{P\overline{C}} = \frac{1}{\cos\theta}$, we see that the pedal $\triangle\overline{ABC}$ of Section 4 is the smallest triangle inscribed in $\triangle ABC$ that is directly similar to $\triangle A'B'C'$ where $\overline{A}, \overline{B}, \overline{C}$ lie on sides BC, CA, AB and with the angles $\overline{A} = A', \overline{B} = B', \overline{C} = C'$.

Since the pedal triangle $\triangle\overline{ABC}$ of P is the strictly smallest triangle inscribed in $\triangle ABC$ that is directly similar to $\triangle A'B'C'$ where $\overline{A}, \overline{B}, \overline{C}$ lie on sides BC, CA, AB with the angles $\overline{A} = A', \overline{B} = B', \overline{C} = C'$, it follows immediately that the point P must be unique since it is impossible to have two strictly smallest triangles $\triangle\overline{ABC}$.

From p. 140, [1], we know that the feet of the three perpendiculars to the sides of a triangle are colinear if and only if P lies on the circumcircle of the triangle. This line is called the Simpson line of P with respect to $\triangle ABC$.

We also note that the medial $\triangle A'B'C'$ of $\triangle ABC$ is the pedal triangle of the circumcenter O of $\triangle ABC$. Also, the vertices A', B', C' of $\triangle A'B'C'$ are listed in counterclockwise order if A, B, C are listed in counterclockwise order. Using a continuity argument with the above fact about Simpson lines, it is easy to see that the pedal triangle $\triangle\overline{ABC}$ of any point P lying inside of the circumcircle of $\triangle ABC$ will have its vertices $\overline{A}, \overline{B}, \overline{C}$ listed in counterclockwise order when $\overline{A}, \overline{B}, \overline{C}$ lie on BC, CA, AB. From the following fact about inverse points of the circumcircle of $\triangle ABC$, it follows that the pedal $\triangle\overline{ABC}$ of any point P lying outside of the circumcircle of $\triangle ABC$ will have its vertices $\overline{A}, \overline{B}, \overline{C}$ listed in clockwise order when $\overline{A}, \overline{B}, \overline{C}$ lie on BC, CA, AB.

From p.173, [1], we know that the pedal triangles for a given triangle of two points P, P' that are inverse for the circumcircle of the given triangle are inversely similar. We say that P, P' are inverse points for the circle (O, R) if P, P', O are colinear, P, P' lie on the same side of O and $OP \cdot OP' = R^2$.

If $\triangle ABC$ is fixed, the pedal triangles of the points in the plane that do not lie on the circumcircle of $\triangle ABC$ give all of the different types of inscribed triangles up to direct and inverse similarity and the rotation of each pedal triangle about each fixed point P gives all of the inscribed triangles that are directly similar to the pedal triangle of P.

It is also fairly obvious that P has the same relation (i.e., P is the same type of point) to all of the triangles $\triangle\overline{A}'\overline{B}'\overline{C}'$ that are obtained by rotating the pedal triangle $\triangle\overline{ABC}$ of P about P. For example if P is the centroid (or the orthocenter or the circumcenter, etc.) of the pedal $\triangle\overline{ABC}$ of P then P is also the centroid (or orthocenter or circumcenter, etc) of all of, the rotated triangles $\triangle\overline{A}'\overline{B}'\overline{C}'$.

6 Some Concluding Remarks

Suppose $\triangle ABC$ is a given triangle and we wish to find all triangles $\triangle\overline{ABC}$ inscribed in $\triangle ABC$ so that $\triangle\overline{ABC}$ is directly similar to $\triangle ABC$ itself with $\overline{A}, \overline{B}, \overline{C}$ on sides BC, CA, AB and angle $\overline{A} = A, \overline{B} = B, \overline{C} = C$.

It is easy to see that the pedal triangle of the circumcenter O of $\triangle ABC$ to $\triangle ABC$ (which is called the medial triangle) is directly similar to $\triangle ABC$. The medial triangle is usually denoted by $\triangle A'B'C'$ and $O = H'$ where H' is the orthocenter of $\triangle A'B'C'$. (The orthocenter H' of $\triangle A'B'C'$ is the intersection of the three altitude of $\triangle A'B'C'$). Thus, it is easy to see that all triangles $\triangle\overline{ABC}$ inscribed in $\triangle ABC$ with $\overline{A}, \overline{B}, \overline{C}$ on BA, CA, AB with angle $A = \overline{A}, B = \overline{B}, C = \overline{C}$ and with $\triangle\overline{ABC}$ directly similar to $\triangle ABC$ are obtained by rotating the medial $\triangle A'B'C'$ about its orthocenter H' and all such triangles $\triangle\overline{ABC}$ have the same orthocenter H'. We also invite the reader to consider inscribed triangles that are directly similar to the orthic triangle, which is the pedal triangle of the orthocenter H.

We mention that the orthocenter H is the incenter of the orthic triangle.

Also, the reader might like to consider the pedal triangles of other points e.g., the Lemoire point K and the Brogard points M, N. Finally, we note that the drawing in Fig. 4 can be generalized by replacing $\triangle ABC$ by any polygon $ABCD\cdots$.

References

[1] Court, Nathan A., College Geometry, Barnes and Noble, New York, 1963.