**Invariant Relations for the Derivatives of Polynomials**

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# 1 Introduction

Using the resultant of two polynomials, we show how to calculate a collection of invariant relations for the derivatives of any polynomial $P(x) = \sum_{i=0}^{n} a_i x^i$.

As a simple example, for the quadratic $Q(x) = a_2x^2 + a_1x + a_0$, we have

$$(Q'(x))^2 - 2Q''(x) \cdot Q(x) = (Q'(0))^2 - 2Q''(0) \cdot Q(0) = a_1^2 - 4a_0a_2.$$

As the degree $n$ of the polynomial $P(x)$ increases, the complexity of most of these invariant relations increases very rapidly. For this reason, the majority these invariant relations are mostly of theoretical interest for large $n$.

# 2 The Resultant of two Polynomials

The resultant $\rho(P(x), Q(x))$ of two polynomials $P, Q$ is a standard determinant which gives by its zero or non-zero value the necessary and sufficient condition so that $P$ and $Q$ have no roots in common. Also, if $P(x) = a_n \cdot \prod_{i=1}^{n} (x - r_i)$ and $Q(x) = b_m \cdot \prod_{i=1}^{m} (x - s_i)$, then $\rho(P(x), Q(x)) = a_n^m \cdot b_m^n \cdot \prod (r_i - s_j)$.

This resultant is the tool that we use in this paper.

**Lemma 1** *Suppose* $P(x) = \sum_{i=0}^{n} a_i x^i$ *and* $Q(x) = \sum_{i=0}^{m} b_i x^i$ *are arbitrary polynomials. Define* $\overline{P}(x) = P(x+b)$ *and* $\overline{Q}(x) = Q(x+b)$. *Then* $\rho\left(\overline{P}(x), \overline{Q}(x)\right) = \rho(P(x), Q(x))$.

**Proof.** Let $r_1, r_2, \cdots, r_n$ be the roots of $P(x)$ and let $s_1, s_2, \cdots, s_m$ be the roots of $Q(x)$. Also, let $\overline{r}_1, \overline{r}_2, \cdots, \overline{r}_n$ be the roots of $\overline{P}(x)$ and let $\overline{s}_1, \overline{s}_2, \cdots, \overline{s}_m$ be the roots of $\overline{Q}(x)$. Now each $\overline{r}_i = r_i - b$ and each $\overline{s}_j = s_j - b$. Also, $\rho(P, Q) = a_n^m \cdot b_m^n \cdot \prod (r_i - s_j)$. Therefore, $\rho\left(\overline{P}, \overline{Q}\right) = a_n^m \cdot b_m^n \cdot \prod (\overline{r}_i - \overline{s}_j) = a_n^m \cdot b_m^n \cdot \prod (r_i - s_j) = \rho(P, Q)$. ■

# 3 Computing Invariant Relations for the Derivatives of a Polynomial

Let us define the polynomial $P(x) = \sum_{i=0}^{n} A_i x^i$ where $A_0, A_1, \cdots, A_n$ are constants. Now $P(x) = \sum_{i=0}^{n} A_i(b)(x-b)^i = \sum_{i=0}^{n} \frac{P^i(b)}{i!}(x-b)^i$ where $P^i(b)$ is the $i$th derivative of $P(x)$ evaluated at $x = b$ and $P^0(b) = P(b)$.

Therefore, $P(x+b) = \sum_{i=0}^{n} A_i(b)x^i = \sum_{i=0}^{n} \frac{P^i(b)}{i!}x^i$. Thus, for all $i \in \{0, 1, 2, \cdots, n\}$, $A_i(b) = \frac{P^i(b)}{i!}$. Also, by letting $b = 0$, we see that for all $i \in \{0, 1, 2, \cdots, n\}$, $A_i = A_i(0) = \frac{P^i(0)}{i!}$ since $P(x) = \sum_{i=0}^{n} A_i x^i = \sum_{i=0}^{n} A_i(0)x^i$.

Let us call $Q(x) = P(x+b) = \sum_{i=0}^{n} A_i(b)x^i = \sum_{i=0}^{n} \frac{P^i(b)}{i!}x^i$.

From Lemma 1, we see that if $P^i(x), P^j(x)$ are the $i$th, $j$th derivatives of $P(x)$ and $Q^i(x), Q^j(x)$ are the $i$th, $j$th derivatives of $Q(x)$, including $P^0 = P, Q^0 = Q$, then $\rho(P^i(x), P^j(x)) = \rho(Q^i(x), Q^j(x))$. This follows since $Q^i(x) = (P(x+b))^i = P^i(x+b)$ and $Q^j(x) = (P(x+b))^j = P^j(x+b)$.

Now $\rho(P^i(x), P^j(x))$ is just an algebraic polynomial expression involving the constants $A_0, A_1, A_2, \cdots, A_n$ where, of course, each $A_i = A_i(0)$.

Also, $\rho(Q^i(x), Q^j(x))$ is the exact same polynomial expression except that each $A_i, i = 0, 1, 2, \cdots, n$, has been replaced by $A_i(b) = \frac{P^i(b)}{i!}$.

Therefore, since $\rho(P^i(x), P^j(x)) = \rho(Q^i(x), Q^j(x))$ for each $i \neq j, i, j \in \{0, 1, 2, \cdots, n-1\}$, we see that for each such $i, j$ we have created a polynomial expression that gives an invariant relation for the derivatives of the polynomial $P(x) = \sum_{i=0}^{n} A_i x^i$. This will become more clear after the illustrations in Section 4.

**Observation 1**. Suppose $C, \overline{C}$ are any non-zero constants. Then $\rho(P^i(x), P^j(x)) =$

$\rho\left(Q^i(x), Q^j(x)\right)$ implies that $\rho\left(CP^i(x), \overline{C}P^j(x)\right) = \rho\left(CQ^i(x), \overline{C}Q^j(x)\right)$.

It is usually more convenient to use this last equality.

## 4 Illustrating the Invariant Relations for Quadratic and Cubic Polynomials

We first define the quadratic $P(x) = \sum_{i=0}^{2} A_i x^i = A_2 x^2 + A_i x + A_0$. Also, $Q(x) = P(x+b) = \sum_{i=0}^{2} A_i(b) x^i = \sum_{i=0}^{2} \frac{P^i(b)}{i!} x^i$.

Now

$$\begin{aligned} A_0(b) &= P^0(b) = P(b) = A_2 b^2 + A_1 b + A_0. \\ A_1(b) &= \frac{P'(b)}{1!} = 2A_2 x + A_1. \\ A_2(b) &= \frac{P''(b)}{2!} = A_2. \end{aligned}$$

Now

$$\begin{aligned} \rho\left(P'(x), P(x)\right) &= \rho\left(2A_2 x + A_1, A_2 x^2 + A_1 x + A_0\right) \\ &= \begin{vmatrix} 2A_2 & A_1 & 0 \\ 0 & 2A_2 & A_1 \\ A_2 & A_1 & A_0 \end{vmatrix} = \begin{vmatrix} 0 & -A_1 & -2A_0 \\ 0 & 2A_2 & A_1 \\ A_2 & A_1 & A_0 \end{vmatrix} \\ &= A_2\left[4A_0A_2 - A_1^2\right]. \end{aligned}$$

We can ignore the $A_2$ that is factored out since $A_2 = A_2(b)$. Therefore, we have the invariant relation $4A_0(b)A_2(b) - (A_1(b))^2 = 4A_0A_2 - A_1^2$ where $A_0 = A_0(0), A_1 = A_1(0)$, and $A_2 = A_2(0)$. Of course, this implies that

$$\begin{aligned} \frac{4P(b)P''(b)}{2} - \left(P'(b)\right)^2 &= 2P(b)P''(b) - \left(P'(b)\right)^2 \\ &= 2P(0)P''(0) - \left(P'(0)\right)^2. \end{aligned}$$

This is the same invariant relation that we gave in the Abstract except that we called $P = Q$ and called $b = x$.

Next, we define the cubic polynomial $P(x) = \sum_{i=0}^{3} A_i x^i = A_3 x^3 + A_2 x^2 + A_1 x + A_0$.

Also, $Q(x) = P(x+b) = \sum_{i=0}^{3} A_i(b)\, x^i = \sum_{i=0}^{3} \frac{P^i(b)}{i!} x^i.$

Now

$$\begin{aligned}
A_0(b) &= P^0(b) = P(b) = A_3 b^3 + A_2 b^2 + A_1 b + A_0. \\
A_1(b) &= \frac{P'(b)}{1!} = 3A_3 b^2 + 2A_2 b + A_1. \\
A_2(b) &= \frac{P''(b)}{2!} = 3A_3 b + A_2. \\
A_3(b) &= \frac{P'''(b)}{3!} = A_3.
\end{aligned}$$

We will study both $\rho(P, P')$ and $\rho(P, P'')$. Of course, $\rho(P', P'')$ is already taken care of since $P'$ is a quadratic. Also $\rho(P, P''')$ is a degenerate case since $P'''(x) = A_3$ is a constant.

Now $\rho(P, P')$ is just the discriminant of $P(x) = A_3 x^3 + A_2 x^2 + A_1 x + A_0$. We can ignore the $A_3$ term that factors out of this discriminant since $A_3 = A_3(b)$.

Therefore,

$$\rho(P(x), P'(x)) = -27A_0^2 A_3^2 + 18A_0 A_1 A_2 A_3 - 4A_0 A_2^3 - 4A_1^3 A_3 + A_1^2 A_2^2.$$

See p.117, [2] for this standard discriminant of a cubic. Therefore, we have relation

$$\begin{aligned}
& -27A_0(b)^2 A_3(b)^2 + 18A_0(b) A_1(b) A_2(b) A_3(b) \\
& -4A_0(b) A_2(b)^3 - 4A_1(b)^3 A_3(b) + A_1(b)^2 A_2(b)^2 \\
= & -27A_0^2 A_3^2 + 18A_0 A_1 A_2 A_3 - 4A_0 A_2^3 \\
& -4A_1^3 A_3 + A_1^2 A_2^2.
\end{aligned}$$

Substituting

$$\begin{aligned}
A_0(b) &= P(b), A_1(b) = P'(b), A_2(b) = \frac{P''(b)}{2}, A_3(b) = \frac{P'''(b)}{6}, \\
A_0 &= A_0(0) = P(0), A_1 = A_1(0) = P'(0), \\
A_2 &= A_2(0) = \frac{P''(0)}{2}, A_3 = A_3(0) = \frac{P'''(0)}{6}
\end{aligned}$$

gives us one invariant relation involving the derivatives of the cubic $P(x) = A_3 x^3 + A_2 x^2 + A_1 x + A_0$.

Now

$$\begin{aligned}
\rho\left(\frac{P''(x)}{2}, P(x)\right) &= \rho\left(3A_3x + A_2, A_3x^3 + A_2x^2 + A_1x + A_0\right) \\
&= \begin{vmatrix} 3A_3 & A_2 & 0 & 0 \\ 0 & 3A_3 & A_2 & 0 \\ 0 & 0 & 3A_3 & A_2 \\ A_3 & A_2 & A_1 & A_0 \end{vmatrix} \\
&= \begin{vmatrix} 0 & -2A_2 & -3A_1 & -3A_0 \\ 0 & 3A_3 & A_2 & 0 \\ 0 & 0 & 3A_3 & A_2 \\ A_3 & A_2 & A_1 & A_0 \end{vmatrix} \\
&= A_3\left[2A_2^3 - 9A_1A_2A_3 + 27A_0A_3^2\right].
\end{aligned}$$

We again ignore $A_3$ since $A_3 = A_3(b)$.

Therefore, we have the relation

$$\begin{aligned}
& 2A_2(b)^3 - 9A_1(b)A_2(b)A_3(b) + 27A_0(b)A_3(b)^2 \\
= \;& 2A_2^3 - 9A_1A_2A_3 + 27A_0A_3^2.
\end{aligned}$$

Again by substituting $A_0 = P(b), A_1(b) = P'(b), A_2(b) = \frac{P''(b)}{2}, A_3(b) = \frac{P'''(b)}{6}, A_0 = A_0(0) = P(0),\ A_1 = A_1(0) = P'(0), A_2 = A_2(0) = \frac{P''(0)}{2}, A_3 = A_3(0) = \frac{P'''(0)}{6}$, we have a second invariant relation involving the derivatives of the cubic polynomial $P(x) = A_3x^3 + A_2x^2 + A_1x + A_0$. Of course, as stated previously, $\rho(P', P'')$ will fall under the classification of the invariant for quadratic polynomials since $P'(x)$ is a quadratic.

# 5 Discussion

As $n$ gets larger, most of these invariant relations increase very rapidly in complexity. However, one of these invariants, namely $\rho(P, P^{n-1})$, can be computed easily. The reader may enjoy doing this. For example, the reader might like to compute $\rho(P, P''')$ for the fourth degree polynomial. This complexity makes most of these invariant relations mostly of theoretical interest for large $n$.

# References


[1] Barbeau, E. J. Polynomials, (Problem Books in Mathematics) (Paperback), Springer Verlag, New York, 1989.

[2] Weisner, Louis, Introduction to the Theory of Equation, The MacMillan company, New York, 1949.