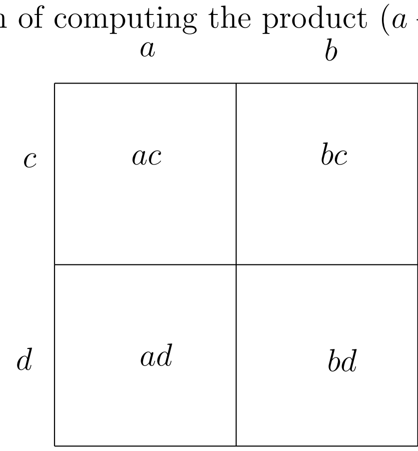
The Area Model for multiplication represents a lovely way to view the distribution property the real number exhibit. This property is the link between addition and multiplication. [1]

1 What is the area model for multiplication?

It is a visual model that represents the product of two sums of numbers as the area of a suitably chosen rectangle. Below are two examples.

1.1 The distributive law

Consider the problem of computing the product $(a+b)(c+d)$.



What this shows is that an $a+b$ by $c+d$ rectangle can be partitioned into four rectangular regions with areas ac, bc, ad and bd, thus proving that $(a+b)(c+d) = ab + bc + ad + bd$. The left side is a product of sums and the right side is a sum of products. This is a good way to think of the area model: A product of sums is a sum of products.

1.2 An Example

	200	10	6
20	4000	200	120
3	600	30	18

Compute the product 216×23 in vertical format. What is the area of the rectangle shown. Find the area of the six rectangles in your work on the product problem.

$$
\begin{array}{r}
216 \\
\underline{\times 23} \\
648 \\
\underline{4320} \\
4968
\end{array}
$$

1.3 Using negative numbers

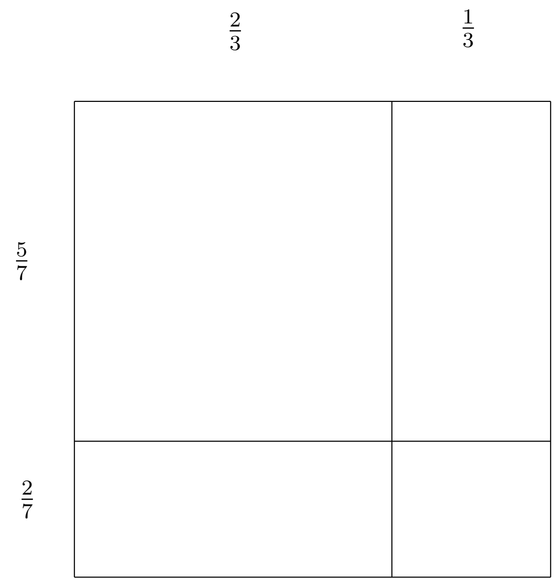
Build a rectangle that measures 216×19 and compute its area.

	200	10	6
20	4000	200	120
-1	-200	-10	-6

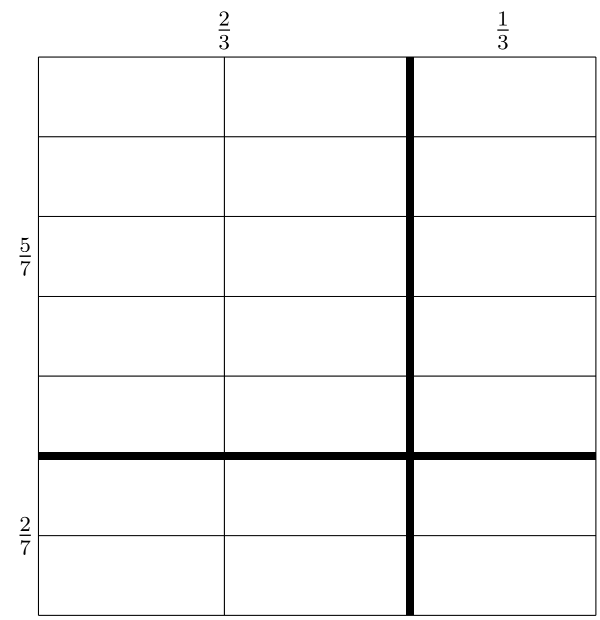
Note that the area model works beautifully with negative numbers. That is because it is really just another way to look at the distribution of multiplication over addition. Thus $216 \times 19 = (200 + 10 + 6)(20 - 1) = 4000 + 200 + 120 - 200 - 10 - 6 = 4104$.

1.4 Multiplying Fractions

The area model works to explain how to multiply two fractions less than 1. Consider the problem $\frac{2}{3} \times \frac{5}{7}$.



Build a grid of 21 congruent rectangles by splitting the vertical segment into 7 equal parts and splitting the horizontal segment into 3 equal parts.

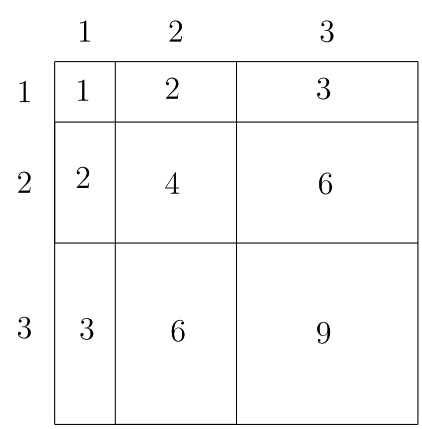


Now each small rectangle represents 1 21$^{\text{st}}$ of the unit square, and the area model shows that $5 \times 2 = 10$ of these represent our product $\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$. Another way to think about this is that $\frac{5}{7}$ of the 21 rectangles is 15 and two-thirds of that is 10 of them.

1.5 One more visual

Have a look at the diagram below. Try to explain it to a friend.

	1	2	3
1	1	2	3
2	2	4	6
3	3	6	9



1.6 Multiplying polynomials

Next, use the area model to compute $(2x^2 + x + 6)(2x + 3)$.

	$2x^2$	x	6
$2x$	$4x^3$	$2x^2$	$12x$
3	$6x^2$	$3x$	18

So, $p(x) = (2x^2+x+6)(2x+3) = 4x^3+2x^2+12x+6x^2+3x+18 = 4x^3+8x^2+15x+18$. You can point out that when $x = 10$, we get just the answer we got above, 4968. Note that the polynomial arithmetic is actually easier than the integer arithmetic since there are no 'carries' to worry about. Now we know that polynomial arithmetic helps us understand integer arithmetic. Can we reverse the process and learn something about polynomials from integers. Of course! Note that the value $p(10) = 4000+800+150+18 = 4968$. Build the factor tree for 4968. You'll see that $4968 = 2^3 \cdot 3^3 \cdot 23$, which we can write as $216 \cdot 23 = 2 \cdot 10^2 + 1 \cdot 10^1 + 6 \cdot 10^0 \cdot 2 \cdot 10^1 + 3 \cdot 10^0$. This suggests the factorization $(2x^2 + x + 6) \cdot (2x + 3)$. Of course this is no surprise.

Here's a practice problem to help you understand what we just saw. Find the prime factorization of $N = 10101$. Then write N as a product of a three-digit number $\underline{abc}$ and a two-digit number $\underline{de}$. Finally, build two quadratic polynomials from the numbers you've found to get a factorization of $x^4 + x^2 + 1$.

1.7 Some Practice Problems.

Use the area model matrix to compute the following products.

1. $(a+b+c)^2$
2. $(a+b-c)^2$
3. $(a+b+c)(a+b-c)$
4. $(a+b+c)(a-b-c)$
5. $(a-b+c)(a+b-c)$
6. $(a^2-2ab+b^2)(a^2+2ab+b^2)$
7. $(a+b)^3$
8. $(a-b)^3$
9. $(a^2-a+1)(a^2+a+1)$
10. $(a+1)^4$
11. $(a+b)(a^2-ab+b^2)$
12. $(a-b)(a^2+ab+b^2)$
13. $(1+x+x^2+x^3)^2$

2 Some Problems with Cubical Dice.

We start this section with some related material. Build the addition and multiplications tables for the set $\{1, 2, 3, 4, 5, 6\}$. If we roll a pair of cubical distinguishable dice and agree to add the results, we can view the sample space (the set of possible outcomes) as the addition table. See the table below.

Use the area model idea to construct the square of the polynomial $p(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$. The matrix shows the number of ways a pair of dice can yield the sums 2 through 12.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

The matrix has 36 equally likely sums, so the probability distribution of the sum of the dice is given in the table below.

2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

2.1 Some Alternative Labelings

Is it possible to label the faces of two cubical dice so that the distribution of sums is just the same as that for the ordinary labeling. The set of problems offered below will enable the reader to find the desired labeling and also to learn about the very nice connection that enables the counting of certain kinds of objects using polynomials. Thus, this essay can be considered an introduction to *generating functions*. A generating function is a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where a_i is the number of objects of type i to be counted. For a regular die, given by the net

	1		
2	3	5	4
	6		

we have six types of objects each occurring once. So the generating function is the polynomial $f(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$. If we had the die labeled

	1		
1	3	5	5
	5		

, we would have the generating function $g(x) = 2x + x^3 + 3x^5$. What's so great about generating functions? Its the way they behave under multiplication. Suppose we want to find the distribution (histogram) of sums of the two dice described above. Notice that the product $f(x)g(x)$ can be computed using the area model we saw above.

	x	x^2	x^3	x^4	x^5	x^6
$2x$	$2x^2$	$2x^3$	$2x^4$	$2x^5$	$2x^6$	$2x^7$
x^3	x^4	x^5	x^6	x^7	x^8	x^9
$3x^5$	$3x^6$	$3x^7$	$3x^8$	$3x^9$	$3x^{10}$	$3x^{11}$

Collecting terms of the same type, we get $f(x)g(x) = 2x^2 + 2x^3 + 3x^4 + 3x^5 + 6x^6 + 6x^7 + 4x^8 + 4x^9 + 3x^{10} + 3x^{11}$. Notice that the sum of the coefficients of the product polynomial is 36, just as we have 36 distinguishable outcomes, assuming we can distinguish the two 1's and the three 5's on the second die.

Our problem can now be stated as follows. Find a pair of different labelings of two dice that have the same probability distribution as that above.

Suppose you have two cubes you'd like to decorate with dots so that when the pair is rolled like dice the probabilities of getting a given sum is exactly the same as the probability would be for a pair of standard dice. To solve this problem consider the *generating function* $f(x)$ of an ordinary die. Its just $f(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$, so we can pretty quickly arrive at $f^2(x) = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}$. If we can factor this into a pair of polynomials each with sum of coefficients different from those of f, we will know how to build the Sicherman dice. We saw above that $x^4 + x^2 + 1$ has a rather nice factorization, namely

$$x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1).$$

Hence we can write $x^6+x^5+x^4+x^3+x^2+x = x^2(x^4+x^2+1)+x(x^4+x^2+1)$.

More generally, a **generating function** is an algebraic device in which a sequence a_n, $n = 0, 1 \ldots$, is coded as an infinite series in a variable x. The algebra of the series can then be used to get useful information about the sequence. Formally, the generating function $f(x)$ of a sequence a_n is the infinite series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots .$$

The sequence a_n is called the **sequence of** $f(x)$.

2.2 Probability Problems with Ordinary Dice.

There are 36 equally likely outcomes. Letting x denote the result of the first die and y the second, we can abbreviate problems like 'what is the probability that the result of the first roll is less than that of the second?' as 'find $P(x < y)$. With this notation in mind, try the 9 problems below. Find the probability of each event.

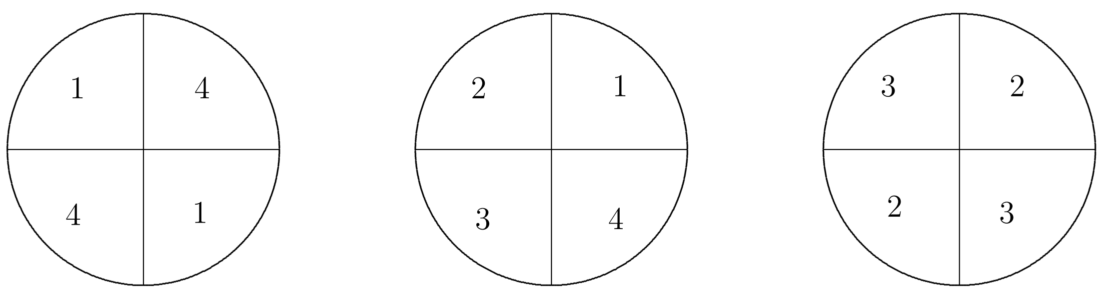
1. $P(x + y \leq 9)$
2. $P(x + y \text{ is even})$
3. $P(x + y \text{ is divisible by } 3)$
4. $P(x + y = 7)$
5. $P(x = y)$
6. $P(x \cdot y \text{ is even})$
7. $P(x \cdot y \text{ is a multiple of } 3)$
8. $P(x^2 + 2xy + y^2 = 12)$
9. Suppose a third die is rolled, with outcome denoted z. Find $P(x + y + z \leq 10$.

2.3 Exercises On Generating Functions.

1. Let $a_n = n$ and $b_n = (-1)^n$. Let these sequences have generating functions $f(x)$ and $g(x)$, respectively. Compute the product $h(x) = f(x)g(x)$ and find the first six terms of $h(x)$.

2. Elizabeth has 3 cubical dice, one numbered $1, 1, 2, 2, 3, 3$, one numbered $2, 2, 4, 4, 6, 6$ and one numbered $1, 1, 3, 3, 5, 5$. She rolls all three simultaneously. What is the probability that the sum of the three faces is odd? What if she rolls two dice of each type?

3. Suppose now that Elizabeth has three spinners like the ones shown below.



Use polynomial multiplication to find the histogram of values of the sum of the three spinners. IE, complete the table below.

4	5	6	7	8	9	10	11

4. Build your own pair of dice with wooden cubes. Use any positive integer values for the faces. The dice can be different. Then build the generating functions for each die and finally compute the product of the two generating functions.

5. The number of two element subsets of a nine element set is $\binom{9}{2} = 36$. This brings up the question, is it possible to label the faces of nine cards with numbers (not necessarily integers) so that when two cards are randomly selected, the probability $p(k)$ that their sum is k is exactly given by

$$p(k) = (6 - |k - 7|) \div 36,$$

for all integers $k \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$?

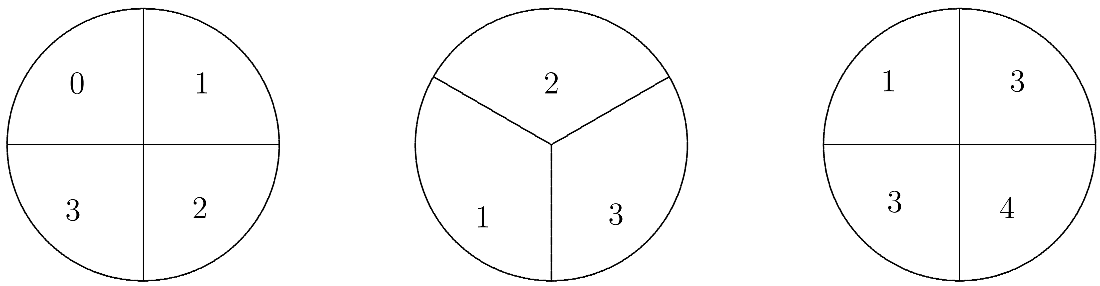
6. There are three urns on a table. One has 2 red balls, one has 2 green balls, and one has 3 blue balls. Balls of the same color are indistinguishable. In how many ways can I choose 4 balls between the urns?

7. Six boys and eight girls go on a nature hike. Each boy will pick either zero or two flowers and each girl will pick either 0 or 3 flowers. How many ways can the group collectively pick 20 flowers?

8. Label the faces of two octahedral dice with the numbers 1 to 8. Then build the polynomial $p(x)$ that describes a single die. Then construct $p^2(x)$ to get the generating function of the sum of the faces of the two dice. Next, go through the factoring process like we did for cubical dice to find some alternative ways to label the octahedra so that you still get $p^2(x)$ as the generating function. Find the number of ways to label the two octahedral dice with non-negative integers so that the distribution of sums is the same as for the ordinary labeling. Alternatively, suppose we are allowed three spinners. How many ways can three spinners be built.

9. Create three spinners which together simulate the rolling a pair of dice.

10. Find the generating function for the sum of the three spinners shown below.



Use polynomial multiplication to find the histogram of values of the sum of the three spinners. IE, complete the table below. **Show your work.**

2	3	4	5	6	7	8	9	10

Letting x, y and z denote the values of the first second and third spinners, find the probability that $x + y + z$ is an even number.

3 A mathematical challenge

The numbers $2, 4, 5, 6, 8, 9$ are arranged in a multiplication table using each digit exactly once. The table can be $1 \times 5, 2 \times 4$ or 3×3. The number of entries is either 5, 8 or 9. Such tables could look like

, or

, or

.

In case it is 3×3, it would look like the table below where each letter represents a different digit. The multiplication table is completed and the sum of the entries is tabulated. What is the largest possible sum obtainable.

$\times$	a	b	c
d			
e			
f			

4 Challenges for middle school students

1. Consider the multiplication table for nonzero digits below.

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

(a) Find the number of even entries and the sum of the even entries in the table.

(b) Find the number of the multiples of 3 in the table and their sum.

(c) Find the number of multiples of 4 and the sum of the multiples of 4 among the 81 products in the table.

(d) Find the sum of the multiples of 6 in the table.

(e) Now let p be an odd prime number. Consider the p^2 by p^2 array of products obtained by listing the numbers from 1 to p^2 along the top and down the column. The table above is what you get for $p = 3$. There are p^4 entries in this array.

i. How many of the p^4 entries are multiples of p?

ii. What is the sum of all the entries that are multiples of p?

(f) Let N denote the product of the 81 entries in the table. Express N in the form $(a!)^b$ for positive integers a and b. Let M be the product of the 56 even entries. Express M in the form $2^a(9!)^b(4!)^c$.

2. This problem is about multiplicative palindromes.

   Notice that
   $$46 \times 96 = 69 \times 64$$
   and
   $$24 \times 63 = 36 \times 42.$$
   Find all pairs of two-digit numbers ab and cd, such that
   $$ab \times cd = dc \times ba.$$
   Let us agree to arrange the four numbers so that $a \leq b$ and ab is the least of the four numbers. We insist that there be at least three distinct digits, so we don't allow $11 \times 22 = 22 \times 11$, for example.

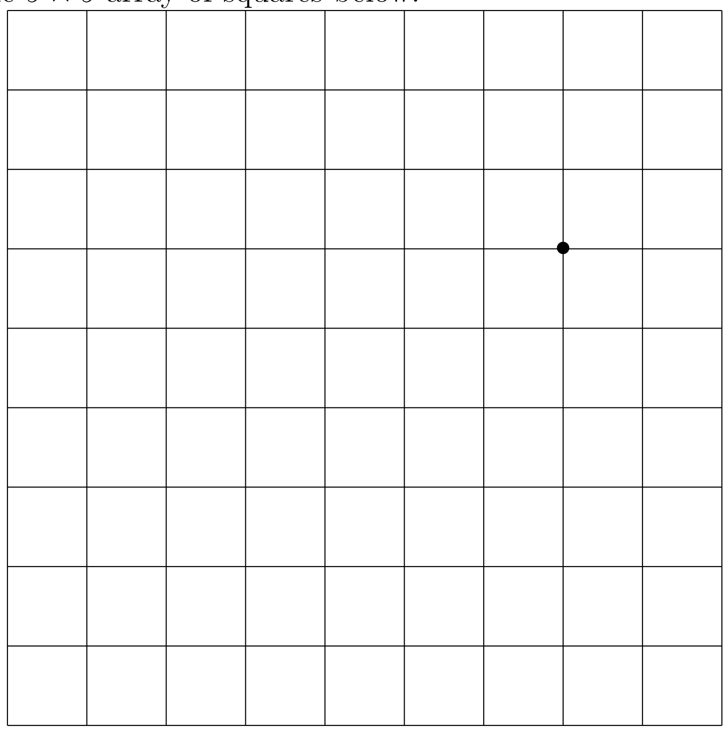
3. Note also that we can build longer multiplicative palindromes.
   $$3516 \times 8274 = 4728 \times 6153$$
   and
   $$992 \times 483 \times 156 = 651 \times 384 \times 299.$$
   Can you find more of these?

4. Pythagorean Palindromes. Note that
   $$14^2 + 87^2 = 78^2 + 41^2.$$
   Find all the two digit numbers $\underline{ab}$ $a \neq b$ for which there exists a two digit number $\underline{cd}$ such that a, b, c, d has at least three different digits and
   $$ab^2 + cd^2 = dc^2 + ba^2.$$

5. Using all the digits $2, 4, 6, 8, 5, 9$ build two three digit numbers whose product is as large as possible. What is the smallest possible product?

6. Using all the digits $1, 3, 5, 7, 9$ each exactly once build two numbers whose product is as large as possible. What is the smallest possible product?

7. Consider the 9×9 array of squares below.



Find the number of rectangular regions whose lower right corner is the bullet. Use this idea to find the total number of rectangular regions in the grid.

How many right triangles can be drawn by just drawing the hypotenuse?

8. Three positive integers a, b and c satisfy both $a \leq b \leq c$ and

$$abc + ab + ac + bc + a + b + c = 71.$$

What is the smallest possible value of $a + b + c$? Next change the problem so that a, b and c are not required to be integers. Next change the problem so that $0 < a \leq b \leq c$

9. The sum S of the divisors of 10 is $S = 1+2+5+10 = 18$. The product P of the divisors of 10 is $P = 1 \cdot 2 \cdot 5 \cdot 10 = 100$. Find both the sum and the product of the divisors of a hundred, a thousand, a million, a billion, and a trillion. Hint: ask yourself if the sum S can be realized as the area of a rectangle.

10. If n is a single digit number, define $P(n)$ to be n. If n has more than one digit, let $P(n)$ be the product of n's digits. So. for example $P(50) = 0$ and $P(56) = 30$. Compute the sum $P(10) + P(11) + \cdots + P(99)$. This is a warm-up problem for number 12 below.

11. Suppose we roll a pair of fair dice. These problems are similar to the ones in section 2.

    (a) What is the probability that the sum is even?

    (b) What is the probability that the sum is a multiple of 3?

    (c) What is the probability that the product is even?

    Suppose we roll three fair dice.

    (a) What is the probability that the sum is even?

    (b) What is the probability that the sum is a multiple of 3?

    (c) What is the probability that the product is even?

    Now suppose we roll five fair dice.

    (a) What is the probability that the sum is even?

    (b) What is the probability that the sum is a multiple of 3?

    (c) What is the probability that the sum is at least 18.

    (d) What is the probability that the product is even?

    (e) What is the probability that the product is a multiple of 3?

    (f) What is the probability that the product is a multiple of 5?

5 Challenges for advanced students

12. Consider the $3 \times 4 \times 5$ rectangular block of 60 cubes. How many rectangular subblocks are there? This hard problem might become more accessible if you first solve it for $2 \times 2 \times 2$ and $3 \times 3 \times 3$ blocks. The answer for the $2 \times 2 \times 2$ is 27.

13. Given a positive integer n, let $p(n)$ be the product of the non-zero digits of n. (If n has only one non-zero digit, then $p(n) = n$.) Let $S = p(1) + p(2) + p(3) + \cdots + p(999)$. What is the largest prime factor of S?

14. Let S be the set of five prime numbers $\{2, 3, 5, 7, 11\}$. For each non-empty subset T of S, let $P(T)$ denote the product of the members of T. In case T has just one element, $P(T)$ is that number. Find the sum of all 31 $P(T)$.

6 Factoring

Factoring, or factorizing as it is called in some countries, is the undoing of multiplication. You saw in the problem on multiplication palindromes that it is sometimes much easier to prove that two product of two numbers (polynomials) can be proven to be equal using factoring rather than multiplication. For example, suppose we are asked to factor $4x^3 + 8x^2 + 15x + 18$ as a product of two polynomials. One reason to learn to factor polynomials is to find the solution set in the plane of equations of the form $f(x, y) = 0$ where f is a polynomial in two variables. Exercises. In the cases with just a single variable, try using the factorization of the number $f(10)$ to find your polynomial factorization. For each of the polynomial functions given below, find a factorization and then use it to solve $f(x, y) = 0$. In cases where the set of points $\{(x, y)|\ f(x, y) \neq 0\}$ is not connected, find the number of regions into which the plane is split by $f(x, y) = 0$. For example $y = 2x$ splits the plane into two regions, while $xy = 0$ splits it into four regions. Thanks to Rick Armstrong for some of these problems.

(a) $f(x) = x^6 - 1$. Note that f can be factored in two ways, as a difference of squares and as a difference of cubes. Do these factorizations lead to the same prime factors. Find all solutions to $f(x) = 0$.

(b) $f(x, y) = xy$

(c) $f(x, y) = x^2y^2 - x^6 - y^6 + x^4y^4$

(d) $f(x, y) = (y^2 - x^4)(x^2 - y^4)(x^2 + y^2 - 1)$

(e) $f(x, y) = (xy^3 - yx^3)(x^2y^2 - x^6 - y^6 + x^4y^4)$

(f) $f(x, y) = x^2 - y^2 + 4x + 6y - 5$

(g) $f(x, y) = x^4 - 4y^2 + 2y^+x^2$

(h) $f(x, y) = x^4 - 3x^2y^2 + y^4$

(i) $f(x, y) = x^8 - 18x^6 + 108x^4 - 264x^2 + 192$

(j) $f(x) = x^3 + 9x^2 + 27x + 19$

(k) $f(x, y) = 25x^2 + 9y^2 - 100x + 54y - 44$

(l) $h(x, y) = 125x^3 + 64y^3$

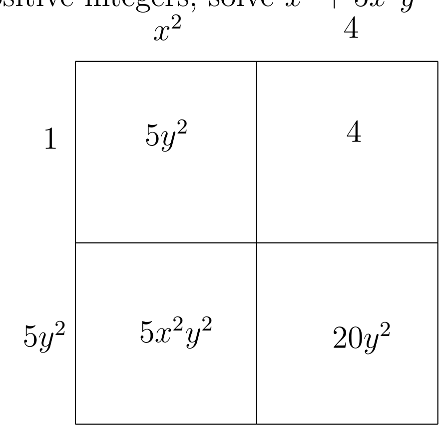
(m) $h(x, y) = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

(n) $h(x) = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32.$

(o) Sometimes you're interested only in integer solutions. If that is the case, solve $x(x + 1)(x + 2)(x + 3) + 1 = 379^2$.

(p) $f(x) = x^5 + x + 1$

15. Simon's Favorite Factoring Trick (SFFT) is a great tool for solving certain math contest problems. Here is an example presented both algebraically and geometrically! No surprise. It refers to the area model. Note that three terms represent the AREA of one of the four rectangles in this diagram. Then, we 'complete the rectangle'. EXAMPLE: Given that x and y are positive integers, solve $x^2 + 5x^2y^2 + 20y^2 = 269$.



So, adding 4 to both sides we have $x^2+5x^2y^2+20y^2+4 = 269+4 = 273$. This leads to $(x^2 + 4)(5y^2 + 1) = 273 = 3 \cdot 7 \cdot 13$ and we see that $x = 3$ and $y = 2$.

16. If p and q are non-zero integers, how many ordered pairs (p, q) satisfy $2pq + 2p + 3q = 18$?

17. Twice the area of a non-square rectangle equals the triple of its perimeter. What is the positive integer area?

18. Compute all integer value of n between 90 and 100 inclusive that cannot be written in the form $n = a+b+ab$, where a and b are positive integers.

19. A, M, and C are positive integers such that $A > M > C$ and $A + M + C = 12$. If $AMC + AM + AC + CM = 71$, what is the maximum possible value of A?

20. If $x^5 + 5x^4 + 10x^3 + 10x^2 – 5x = 9$ and $x \neq –1$, compute the numerical value of $(x + 1)^4$.

21. Find the number of ordered pairs of integers (m, n) for which $mn > 0$ and $m^3 + n^3 + 99mn = 33^3$.

22. The number $(9^6 + 1)$ is the product of three primes. Compute the largest of these 3 primes.

23. Of the integers between 1 and 2310, how many are divisible by exactly three of the five primes $2, 3, 5, 7$, and 11?

24. If x and y are positive integers such that $x^2 = y^2 + 61$, find $x(x + 2) + y(y + 3)$.

25. The graph of $xy + 3x + 2y = 0$ can be produced by translating the graph of $y = k/x$ to the left and down for some constant value k. Find k.

26. Let $f(x) = x^2 + bx + 9$ and $g(x) = x^2 + dx + e$. If $f(x)$ has zeroes r and s, and $g(x)$ has zeroes $–r$ and $–s$, compute the two roots of $f(x) + g(x) = 0$.

27. How many ordered pairs of integers (x, y) with $1 < x < 100$ and $1 < y < 100$ make the quantity $xy − x − y$ a multiple of 5?

28. If three of the roots of $x^4 + ax^2 + bx + c = 0$ are $1, 2,$ and 3, find the value of $a + c$.

29. Suppose x and y are real numbers that satisfy equations $x–y = 1$ and $x^5–y^5 = 2016$. Calculate $\frac{x^5+y^5}{x+y} - (x^4 + y^4)$.

30. How many ordered pairs of positive integers (a, b) are that such that $\frac{1}{a} - \frac{1}{b} = \frac{1}{143}$?

31. Suppose that a, b, c, d are real numbers such that $ab+3a+3b = 216$; $bc+3b+3c = 96$; and $cd+3c+3d = 40$. Determine the maximum possible value of $ad + 3a + 3d$.