Graphs and directed graphs (aka digraphs) play an important role in problem solving. They are used to simplify the data, sifting out unneeded information. There is a beautiful example of this in the solitaire puzzle *instant insanity*. Look it up when you have a chance. Another use is found in the section below. It helps us understand the structure of a mathematical object.

Another lovely application of graphs is the elementary Ramsey result. It goes like this: suppose 6 people attend a party. Prove that there are three people who are either mutual friends or mutual strangers. Sounds hard, right? But if your phrase in the language of graphs, it becomes easy. To do this, color each edge of the complete graph K_6 either red (friends) or blue (strangers). Then prove that for any coloring, there is a monochromatic (ie single color) triangle. We can do this using the pigeon-hole principle. Ask me or your TA for a proof.

0.1 A four dimentional cube graph

Consider the same (lattice/Hasse) diagram for the divisors of 210. We can draw this in several ways. The first one (Fig. 6a) places each divisor of 210 at a level determined by its number of prime divisors. The second one (Fig. 6b) emphasizes the 'degrees of freedom'. These two diagrams are representations of a four dimensional cube, not surprising since the Hasse diagram for D_{30} is a three-dimensional cube. A mathematical way to say the two digraphs are the same is to say they are *isomorphic*. This means that they have the same number of vertices and the same number of edges and that a correspondence between the vertices also serves as a correspondence between the edges. Note that the digraphs in 6a and 6b have the required number of vertices (16) and the required number of edges (32). Can you find an N such that the Hasse diagram of D_N is a representation of a five-dimensional cube. Such a digraph must have $2^5 = 32$ vertices, and $2 \cdot 32 + 16 = 80$ edges.

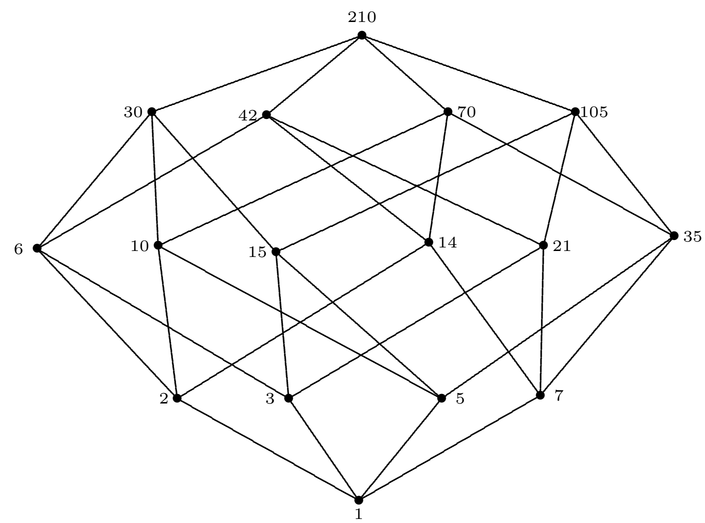


Fig 6a. The Hasse diagram of D_{210}

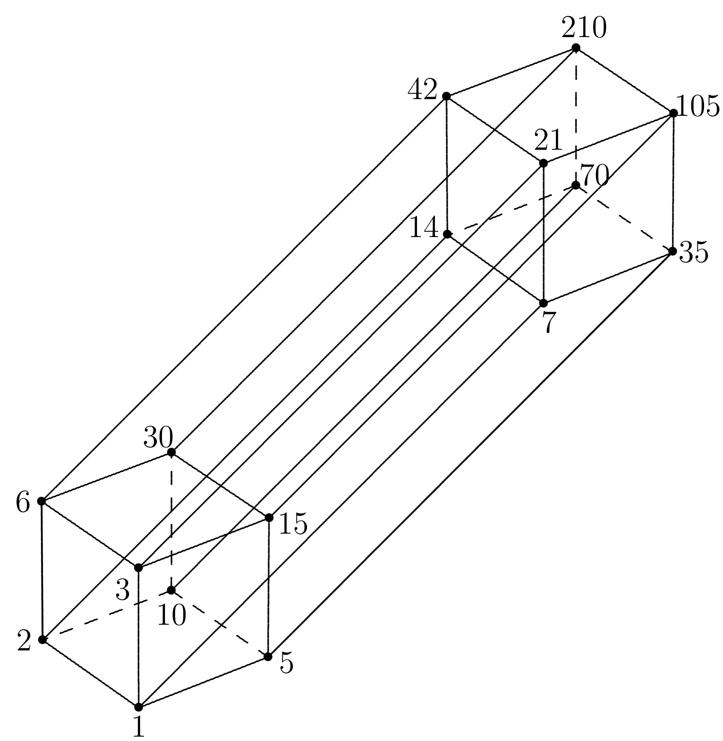
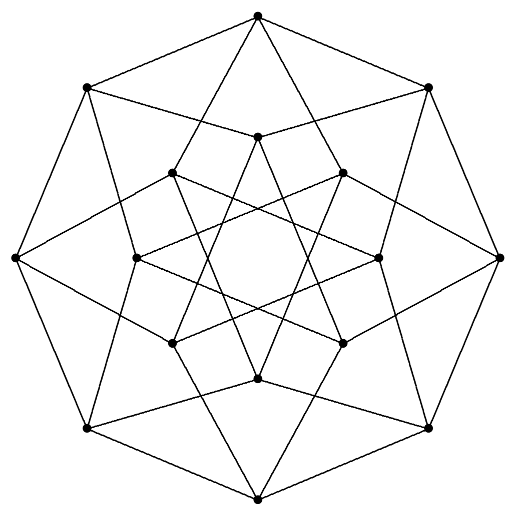
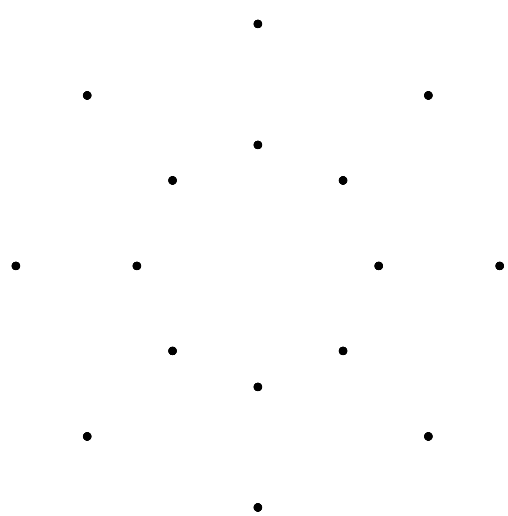


Fig 6b. The Hasse diagram of D_{210}

Notice that the graph below has 16 vertices just as D_{210} and that, like D_{210}, each vertex has degree 4. Is it the same as D_{210}?



Using the dots (vertices) below, try drawing the graph without lifting your pencil. Graphs of this type are called *Eulerian*.



0.2 Other problems with graphs

1. The integers $3, 4, 5, 6, 12$, and 13 are arranged, without repetition, in a horizontal row so that the sum of any two numbers in adjoining positions is a perfect square (the square of an integer). Find the sum of the first and last.

   Solution: 11. Build a graph with the six vertices labeled $3, 4, 5, 6, 12$, and 13. Join two when their sum is a perfect square. Note that 5 and 6 are related to only 4 and 3 respectively, so they must be at the ends. The sequence 6 3 13 12 4 5 works.

2. Find the largest positive integer with the following properties : (a) all the digits are different. (b) each two consecutive digits form a number divisible by either 17 or 23.

   Solution: 923468517.

3. A six-digit positive integer is *sixish* if it has six different non-zero digits and the product of each pair of adjacent digits is divisible by 6. How many sixish numbers are there?

   Solution: 216. We can build a graph with 9 vertices, one for each digit. It's something like a wheel with spokes: 6 in the middle connected to all the digits. Then it has a bipartitie looking subgraph with even digits (not 6) connected to 3, 6, and 9. I start by asking, what if the number starts or ends with a 1, 5 or 7? Then what if it start with an even digit 2,4 or 8 and then consider the ones starting with 3,6 or 9.

4. A tennis club has six players numbered 1 to 6. Two of them are to be chosen to represent the club in a tournament. Five assistant coaches make the following recommendations: #4 and #5, #3 and #6, #5 and #6, #2 and #5 and #1 and #3. The head coach ignores both players recommended by one of the assistant coaches, and chooses exactly one player from the recommendation of each of the other four. Which are the two chosen players?

   Solution: $\#1, \#5$. Build the graph. You can see that if 5 is not chosen then two of the assistants will not get either of their choices. If we pick

5 and 3, all the assistants will get one of their choices. So 1 and 5 are the only two players that meet the requirements.

5. The numbers 2, 3, 4, 5, 6, 7, 8 and 9 are to be put into four pairs such that the sum of the two numbers in each pair is a prime. How many different ways are there to split these numbers into four pairs?

   Solution: 6. Build the bipartite graph and notice that the digits 6 and 7 both have degree 2 while all the other vertices have degree 3. Build the trees starting with the pairing of 6 and 7. This leads to four partitions. Then the pairing of 7 with 4 leads to two more, for a total of 6.

6. A 2017-digit number begins with 3. The number formed by any two adjacent digits is either divisible by 17 or 23. There are exactly two such integers that can be formed. What is the positive difference between them?

   Solution: 717. Draw the graph. The only cycle has length 5: $3 \Longrightarrow 4 \Longrightarrow 6 \Longrightarrow 9 \Longrightarrow 2$, so the numbers could end with 69234 or 68517 whose difference is 717.

7. There are four elevators in a building. Each makes three stops, which do not have to be on consecutive floors or include the main floor. For any two floors, there is at least one elevator which stops on both of them. What is the maximum number of floors in this building?

   Solution: 5. Let the floors be letters. With 5 floors, we get letters a, b, c, d, e. Each elevator contributes 3 labeled edges to the complete graph on the 5 vertices a, b, c, d, e, which has 10 edges. The four elevators together contribute 12 labeled edges including all 10 of the graph. On the other hand, four elevators is not enough for 6 floors because the complete graph K_6 have 15 edges.

8. Partition the positive integers from 1 to 30 inclusive into k pairwise disjoint groups such that the sum of two distinct elements in a group is never the square of an integer. What is the minimum value of k?

Solution: 3. Suppose $k = 2$. Then 1 and 3 are in different sets. Now 6 must go with 1, and 15 must go with 3. But now 10 cannot go with 15 or 6. I don't have a partition showing this can done with $k = 3$. In other words the cycle $1, 3, 6, 10, 15$ is an odd length cycle, which cannot be split into two subgraphs using two consecutive vertices. Now for the second part, Selena Ge found $A = \{1, 2, 6, 9, 11, 13, 18, 22, 26, 28, 29\}$, $B = \{3, 4, 7, 15, 17, 20, 23, 24, 27, 30\}$ and $C = \{5, 8, 10, 12, 14, 16, 19, 21, 25\}$. Looking a bit more into the problem, I found that the set $A = \{n|n \equiv 1(\text{mod}3)\}$ works for one of the sets. I used $B = \{n|n \equiv 2(\text{mod}3)\}$ for the second set. It does not work because of the pairs $5, 11; 11, 14; 8, 17; 20, 29$ and $23, 26$. Also the set of multiples of 3 does not quite work either. So I moved the offenders, 6,12 and 21 to set B and moved $11, 17, 20, 23$ to set C to get the solution $A = \{1, 4, 7, 10, 13, 16, 19, 22, 25, 28\}$, $B = \{2, 5, 6, 8, 12, 14, 26, 29, 21\}$, and $C = \{3, 9, 15, 18, 24, 27, 30, 11, 17, 20, 23\}$. Kyle Liao points out that there must be three sets because $6, 19, 30$ must be in different sets. Kyle's solution to this problem results from using the greedy algorithm, starting with $A = \{1, 2, 4, 6, 8, 9, 11, 13, 17, 18, 20, 22, 26, 30\}$, and then building the other two sets again using the same algorithm. The real challenge here is now to find the largest integer N for which the set $1, 2, 3, \ldots, N$ admits a partition into three sets for which no sum in a perfect square, as we required here.

9. The first digit of a positive integer with 2013 digits is 5. Any two adjacent digits form a multiple of either 13 or 27. What is the sum of the different possible values of the last digit of this number?

   Solution: 16 build the digraph to see that only $3, 7, 6$ are last digits.

10. In a chess tournament, each of the 10 players plays each other player exactly once. After some games have been played, it is noticed that among any three players, there are at least two of them who have not yet played each other. What is the maximum number of games played so far?

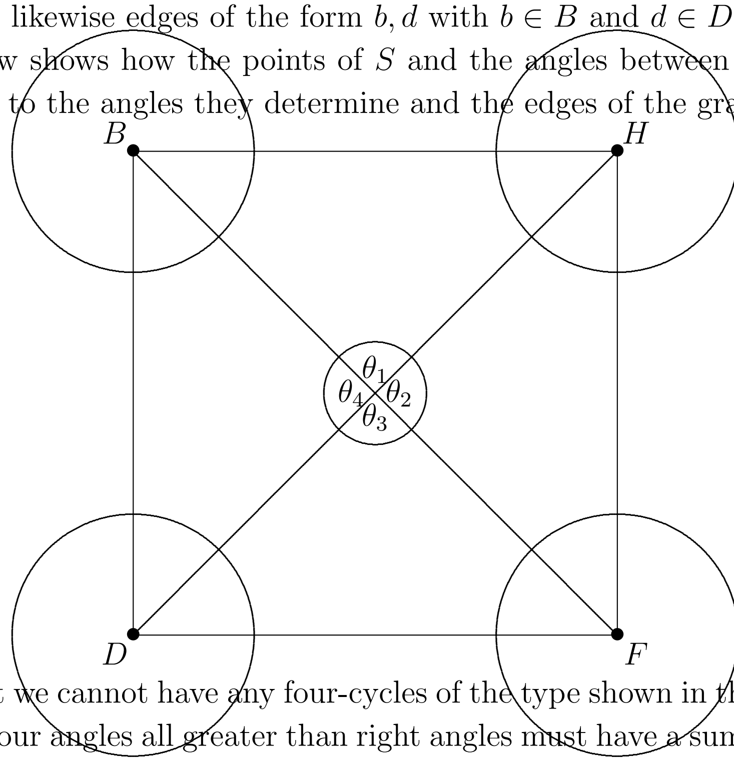
    Solution: 25. Solution by Selena Ge. Let player 1 be the player who has played the most games. Suppose player 1 has played k games.

Then player 1 has not played, let's say, players $k+2, k+3, \ldots 10$. So player 1 has not played $9-k$ others. These $9-k$ others can all play the k players that 1 has plyed without creating a triangle in the graph. This means that at most $k + k(9-k) = k(10-k) = -(k-5)^2 + 25$ games have been played. The maximum value of this is $0^2 + 25 = 25$. We are maximizing the number of edges of a 10 vertex graph subject to the graph not containing any 3-cycles (triangles). We can think of this graph as the complete bipartite graph, $K_{5,5}$. You can see that it has no cycles of length 3.

11. Consider 15 rays that originate from one point. What is the maximum number of obtuse angles they can form? (The angle between any two rays is taken to be less than or equal to 180°.)

    Solution: 75 Put the rays in three groups A, B, and C of 5 so that each each ray in A makes an obtuse angle with all the rays in the other two sets, etc. I do not have a proof that 75 is maximal. From Arthur Holshouser. A graph G has four pairwise disjoint sets of vertices A, B, C and D such that $|A| = a$, $|B| = b$, $|C| = c$ and $|D| = d$ with edges only between A and B, B and C, C and D, and finally, D and A. Let's say that G is *saturated* if each of the of the edge sets is maximal. That is, G has $ab+bc+cd+da$ edges. Note that G has $abcd$ four-cycles. Notice that the removal of an edge from A to B can remove at most cd 4-cycles. Reasoning similarly, each edge removal eliminates at most ab, bc, cd, da 4-cycles. Suppose that $cd = max\{ab, bc, cd, da\}$. This is equivalent to $ab = min\{ab, bc, cd, da\}$ since a and b are smaller that c and d. One way to eliminate all the 4-cycles from G by removing the fewest edges is by removing edges only from the edges joining A and B. Each time an edge is removed from $A-B$, the number of 4-cycles is reduced by cd. Since we start with $abcd$ 4-cycles, and each $A-B$ edge removal reduces the number by cd, we must remove ab such edges. It follows that the fewest edges we can remove so that the remaining graph has no 4-cycles is exactly ab. This results in a saturated tripartite graph.

    To solve the main problem, let S be 15 points on the unit circle. Assume that none of the points lie on either the x- or y-axes, and write $S = A \cup B \cup C \cup D$ where all the points of A lie in quadrant 2, B's points in quadrant 1, C's points in quarant 4 and D's points lie in quadrant 3. Next join any two points of S with an edge if the angle between them lies between $90°$ and $180°$. Of course no edge connects pairs of points from the same letter set A, B, C or D. Assuming no two points of S are antipodal, there must be edges joining all pairs a, c with $a \in A$ and $c \in C$, and likewise edges of the form b, d with $b \in B$ and $d \in D$. The figure below shows how the points of S and the angles between them are related to the angles they determine and the edges of the graph.



Notice that we cannot have any four-cycles of the type shown in the figure, since four angles all greater than right angles must have a sum that exceeds $360°$. Therefore, from problem 2, since $cd = max\{ab, bc, cd, da\}$ and $ab = min\{ab, bc, cd, da\}$ and since the graph in the figure above can have no 4-cycles, an upper bound on the number of edges that the graph can have is $bc+cd+da$. Finally, the upper bound can be realized if we simply replace the sets A and B with $A \cup B$. Since there are no edges in $A \cup B$, or in C or in D, the graph is tripartite. At this point we can say that the maximum number of obtuse angles is the number of edges of some tripartite graph with 15 vertices. Now in case we start with n points of the circle, we can find the maximum by letting the three sets be as close as possible in cardinality. So each set should have a number of elements t that satisfies $\lfloor t \rfloor = (n - n(mod 3)) \div 3$.

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12. How many four digit number have the property that each pair of consecutive digits represents a two-digit number that is a multiple of 3. What if the digits have to be distinct?

    Solution: